Lecture 30 : Euler's Theorem

You may have noticed a strange relationship between the Taylor polynomials of cos, sin, and exp. Recall that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + o(x^8)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + o(x^8)$$

while
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + o(x^8).$$

If we substitute ix ($i = \sqrt{-1}$) in for x and assume the $o(x^8)$ remains valid, then we get

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + o(x^8)$$

$$= \cos(x) + i\sin(x) + o(x^8).$$

This at least suggests that, were we to be able to define complex-valued functions of a complex variable by power series, we could expect to get

$$(*) \quad \boxed{e^{ix} = \cos(x) + i \cdot \sin(x)} \qquad \left(\text{e.g.} \overset{}{\underset{x=\pi}{\Longrightarrow}} e^{i\pi} = -1 \right)$$

as a theorem. This observation, due to Euler, is consistent with the fact that

$$(**) \quad \big(\cos(x) + i\sin(x)\big) \cdot \big(\cos(y) + i\sin(y)\big) = \cos(x+y) + i\sin(x+y),$$

proved in the last lecture as a consequence of angle-addition formulas.

Apostol, who takes (for $z = x+iy$)

Still never zero! (why?)

$$e^z := e^x e^{iy} := e^x(\cos(y) + i\cdot\sin(y))$$

(& thus (*)) as a definition, motivates it in a different way: if we are to have $e^{iy} = F(y) + iG(y)$ with F, G twice differentiable, then we should have

$$ie^{iy} = F'(y) + iG'(y) \quad \text{and} \quad -e^{iy} = F''(y) + iG''(y)$$

$$\Rightarrow \begin{cases} F''(y) = -F(y) \\ F(0) = 1,\ F'(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} G''(y) = -G(y) \\ G(0) = 0,\ G'(0) = 1 \end{cases}, \quad \text{which}$$

have (we now know unique) solutions $\cos(y)$ & $\sin(y)$.

As an immediate consequence of this definition and (**), we have

$$e^{z_1+z_2} = e^{z_1}\cdot e^{z_2} \qquad (z_1, z_2 \in \mathbb{C}).$$

Moreover, the "polar form" of a complex number simplifies to

$$\alpha = r(\cos\theta + i\sin\theta) = re^{i\theta} \left(= |\alpha| e^{i\arg(\alpha)}\right).$$

We can also define derivatives & integrals of complex functions of a real variable

$$f(x) = u(x) + iv(x)$$

simply by differentiating & integrating the real & imaginary parts:

$$f'(x) = u'(x) + iv'(x) \quad \text{and} \quad \int_a^x f(t)\,dt = \int_a^x u(t)\,dt + i\int_a^x v(t)\,dt.$$

Ex/ If $f(x) = e^{\alpha x}$, where $\alpha = a+ib \in \mathbb{C}$, then

$$f(x) = e^{ax}e^{ibx} = e^{ax}\cos bx + ie^{ax}\sin bx$$

$$\Rightarrow \quad \underbrace{f'(x)} = a e^{ax}\cos bx - b e^{ax}\sin bx + i\left[a e^{ax}\sin bx + b e^{ax}\cos bx\right]$$

$$= a e^{ax}(\cos bx + i\sin bx) + ib e^{ax}(\cos bx + i\sin bx)$$

$$= \alpha e^{ax}(\cos bx + i\sin bx) = \alpha e^{ax} e^{ibx} = \underbrace{\alpha e^{\alpha x}}. \quad //$$

As a consequence, we get a much simpler description of solutions $y = f(x)$ to

$$(\ast\ast\ast) \qquad y'' + ay' + by = 0.$$

Let α be a root of $\underbrace{r^2 + ar + b = 0}_{\text{characteristic equation}}$; then writing $g(x) = e^{\alpha x}$ we get

$$g''(x) + a g'(x) + b g(x) = \alpha^2 e^{\alpha x} + a\alpha e^{\alpha x} + b e^{\alpha x}$$

$$= \underbrace{(\alpha^2 + a\alpha + b)}_{=0} e^{\alpha x} = 0.$$

If the characteristic equation factors as $(r-\alpha)(r-\bar{\alpha})$, † then the general solution of $(\ast\ast\ast)$ becomes

$$y = f(x) = \gamma_1 e^{\alpha x} + \gamma_2 e^{\bar{\alpha} x}, \quad \gamma_1, \gamma_2 \in \mathbb{C}$$

$$= \gamma_1 e^{-\frac{a}{2}x} e^{i\delta x} + \gamma_2 e^{-\frac{a}{2}x} e^{-i\delta x}$$

$\alpha = -\frac{a}{2} + i\delta$

$$= e^{-\frac{a}{2}x}\left(\gamma_1 e^{i\delta x} + \gamma_2 e^{-i\delta x}\right)$$

$$= e^{-\frac{a}{2}x}\left((\gamma_1+\gamma_2)\cos(\delta x) + i(\gamma_1-\gamma_2)\sin(\delta x)\right).$$

Solving $\gamma_1 + \gamma_2 = c_1$, $i(\gamma_1 - \gamma_2) = c_2$ $(c_1, c_2 \in \mathbb{R})$ then recovers all real solutions to $(\ast\ast\ast)$.

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† If $a, b \in \mathbb{R}$ then the other root is the conjugate of α; in fact, $(r-\alpha)(r-\bar{\alpha}) = r^2 - 2\mathrm{Re}(\alpha) r + |\alpha|^2 = r^2 + ar + b \Rightarrow \alpha = \frac{-a}{2} + i\delta$, where $\delta = \frac{\sqrt{4b - a^2}}{2}$.

Powers of a complex number: writing $\alpha = re^{i\theta}$ in polar form, we have $\alpha^n = r^n (e^{i\theta})^n = r^n e^{in\theta}$. The last step here is by induction: $A(n)$ means $(e^{i\theta})^n = e^{in\theta}$; $A(1)$ is obvious, and $A(n-1) \Rightarrow A(n)$ by $(e^{i\theta})^n = e^{i\theta}(e^{i\theta})^{n-1} = e^{i\theta}e^{i(n-1)\theta} = e^{i[\theta+(n-1)\theta]} = e^{in\theta}$.

Notice that this says $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, which is called de Moivre's Theorem.

Roots of unity are solutions to $z^n = 1$ ("unity" means 1).

There are n of these, $z = e^{i\cdot\frac{2\pi k}{n}}$ $(k = 0, 1, \ldots, n-1)$.

Sometimes one writes $\zeta_n = e^{\frac{2\pi i}{n}}$ so that the n^{th} roots are ζ_n^k.

Ex/ Find the n^{th} roots in Cartesian form $(a+bi)$ for $n = 2, 3, 4, \& 6$. [Challenge: can you do $n=5$?] //

Ex/ Find $1 + \zeta_n + \zeta_n^2 + \cdots + \zeta_n^{n-1}$. Interpret this geometrically. //

Complex logarithm. Given $\alpha = re^{i\theta}$, we may write

$$\alpha = e^{\log r} e^{i\theta} = e^{\log r + i\theta} = e^{\log|\alpha| + i\arg(\alpha)}.$$

For this reason, it is natural to define (so that $e^{\log(\alpha)} = \alpha$)

$$\log(\alpha) := \log|\alpha| + i\arg(\alpha).$$

this part only defined "up to $2\pi i$"

This gives $\log(-1) = \pi i$, $\log(i) = \frac{\pi i}{2}$, etc. and has the property that $\log(\alpha\beta) = \log(\alpha) + \log(\beta)$ "mod $2\pi i$".

That's all for now on complex numbers — more on ODEs Wed.