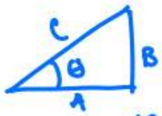
Lecture 31 : More on 2^{nd}-order linear ODEs

Let's begin with some examples of applications, arising from

vibrations —

- undamped case : $\boxed{y'' + k^2 y = 0}$ — think of spring, door on hinge (both w/o friction), electrical circuit w/ inductor & capacitor only, etc.



$$\Rightarrow y = A \sin kt + B \cos kt$$

$\theta := \arctan(B/A)$
$C := \sqrt{A^2+B^2}$

$$= C\{\cos\theta \sin kt + \sin\theta \cos kt\} = \underset{\text{amplitude}}{C} \sin(kt + \underset{\text{initial phase}}{\theta})$$

- damped case : $(c > 0)$ $\boxed{y'' + 2cy' + k^2 y = 0}$ discriminant $d = 4(c^2 - k^2)$
  - $d = 0$ (critical damping) : $y = e^{-ct}(A + Bt)$
  - $d > 0$ (overcritical damping) : $y = Ae^{-[c-\sqrt{c^2-k^2}]t} + Be^{-[c+\sqrt{c^2-k^2}]t}$
  - $d < 0$ (undercritical damping) : $y = e^{-ct}\left(A\sin(\sqrt{k^2-c^2}\,t) + B\cos(\sqrt{k^2-c^2}\,t)\right)$
    ↳ allows some oscillation

- damped + driving force : $\boxed{y'' + 2cy' + k^2 y = F(t)}$ $(*)$

  We don't know how to solve these yet. One place they arise is in simple electrical circuits with an inductor, a resistor, a capacitor & a voltage source : then the current $I(t)$ satisfies

$$LI' + RI + \frac{1}{C}\int I\,du = V(t)$$

$$\Rightarrow LI'' + RI' + \frac{1}{C} I = V'(t).$$

Ex/ A door on a hinge satisfies $\theta'' = \underbrace{-2\theta}_{\text{spring mechanism on hinge}} \underbrace{-3\theta'}_{\text{air resistance}}$.

Write $\omega = \theta'$ for angular velocity. For what initial values θ_0, ω_0 does the door "slam", i.e. have $\omega < 0$ at $\theta = 0$? //

The solution of the <u>inhomogeneous</u> equation (*) uses a beautiful trick involving the Wronskian. Recall that if the general solution of the <u>homogeneous</u> equation $y'' + 2cy' + k^2 y = 0$ is $y = c_1 v_1(t) + c_2 v_2(t)$, and $f(t)$ is one solution of (*), then the general solution of (*) is $y = f(t) + c_1 v_1(t) + c_2 v_2(t)$.

So how to find one solution $f(t)$?

Write $L := \frac{d^2}{dt^2} + 2c\frac{d}{dt} + k^2$ for the linear differential operator so that (*) becomes

$$L y = F,$$

and consider a function of the form $f(t) := u_1(t)v_1(t) + u_2(t)v_2(t)$.

Note that
$$f'(t) = (u_1'v_1 + u_2'v_2) + u_1 v_1' + u_2 v_2'$$
$$f''(t) = (u_1'v_1 + u_2'v_2)' + u_1'v_1' + u_2'v_2' + u_1 v_1'' + u_2 v_2''$$

$$\Rightarrow Lf = f'' + 2cf' + f$$
$$= u_1 \cancelto{0}{(v_1'' + 2cv_1' + v_1)} + u_2 \cancelto{0}{(v_2'' + 2cv_2' + v_2)} \quad \text{(why?)}$$
$$+ (u_1'v_1' + u_2'v_2') + (u_1'v_1 + u_2'v_2)' + 2c(u_1'v_1 + u_2'v_2)$$

$\Rightarrow$ get $Lf = F$ if
$$\begin{cases} u_1'v_1 + u_2'v_2 \equiv 0 \\ u_1'v_1' + u_2'v_2' = F. \end{cases}$$

mult. 1st eqn. by v_1', 2nd by v_1

$$u_1'v_1v_1' + u_2'v_2v_1' = 0$$
$$-(u_1'v_1'v_1 + u_2'v_2'v_1 = v_1F)$$
$$u_2'(v_2v_1' - v_2'v_1) = -v_1F$$
$$u_2' = v_1F/W$$

mult. 1st eqn. by v_2', 2nd by v_2

$$u_1'v_1v_2' + u_2'v_2v_2' = 0$$
$$-(u_1'v_1'v_2 + u_2'v_2'v_2 = v_2F)$$
$$u_1'(v_1v_2' - v_2v_1') = -v_2F$$
$$u_1' = -v_2F/W$$

where $W = v_1 v_2' - v_2 v_1'$. We therefore have

$$(**) \quad \boxed{f(t) = -v_1 \int \frac{v_2 F}{W}\,dt + v_2 \int \frac{v_1 F}{W}\,dt}$$

solves $(*)$. (There are special cases where less effort is required to find f, like if F is a polynomial or e^{rx} times a polynomial. See § 8.16 of Apostol.)

Ex/ $y'' + y' - 2y = e^t + e^{2t}$

First note that the homogeneous equation has characteristic eqn $0 = r^2 + r - 2 = (r+2)(r-1) \Rightarrow v_1 = e^t, \; v_2 = e^{-2t}$.

Now $W = v_1 v_2' - v_2 v_1' = -2e^t e^{-2t} - e^{-2t} e^t = -3e^{-t} \Rightarrow$

$(**)$ yields $f(t) = -e^t \int \frac{e^{-2t}(e^t + e^{2t})}{-3e^{-t}}\,dt + e^{-2t} \int \frac{e^t(e^t + e^{2t})}{-3e^{-t}}\,dt$

$= \frac{1}{3} e^t \int (1 + e^t)\,dt - \frac{1}{3} e^{-2t} \int (e^{3t} + e^{4t})\,dt = \frac{1}{3} e^t (t + e^t) -$

$\frac{1}{3} e^{-2t} \left(\frac{1}{3} e^{3t} + \frac{1}{4} e^{4t}\right) = \left(\frac{t}{3} + \frac{1}{9}\right) e^t + \frac{1}{4} e^{2t}$. //