Lecture 32: Nonlinear first-order equations

Technique #1: $\boxed{\text{Separable equations}}$

These are equations of the form

$$y' = Q(x)\,R(y).$$

Claim: We can just write

$$\frac{dy}{dx} = Q(x)\,R(y) \rightsquigarrow \frac{dy}{R(y)} = Q(x)\,dx \rightsquigarrow \int \frac{dy}{R(y)} = \int Q(x)\,dx$$

(up to constant).

Proof: This requires justification since we don't prove things using "differentials". But the justification is easy: using $y = f(x)$, we write the equation as $f'(x) = Q(x)\,R(f(x))$, or $\dfrac{f'(x)}{R(f(x))} = Q(x)$. Integrating both sides (w.r.t. x) gives

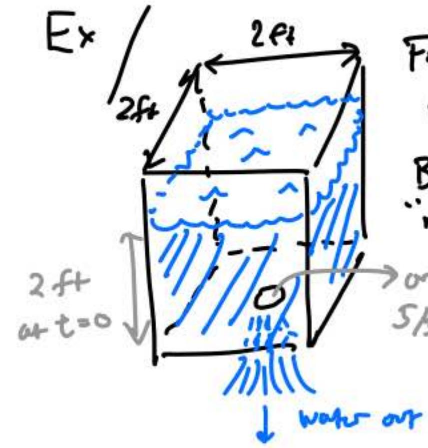
$$\int Q(x)\,dx = \int \frac{f'(x)\,dx}{R(f(x))} \overset{\text{subst. } y=f(x)}{=} \int \frac{dy}{R(y)}. \qquad \square$$

Ex/ $y' + 2xe^{y} = 0 \rightsquigarrow -\int e^{-y}\,dy = \int 2x\,dx$

$$\rightsquigarrow e^{-y} = x^2 + C$$

$$\rightsquigarrow y = -\log(x^2 + C) \qquad //$$

Ex/



Find the time required for the water level to drop to 1 foot? to empty? (Let $y = f(t) =$ level)

Begin by observing that water exiting the orifice is really being "removed" from the top of the water level in the tank — i.e. a water droplet of mass m has "lost" mgy units of potential energy, so upon exit must have kinetic energy $\frac{1}{2}mv^2 = mgy \Rightarrow \underset{\text{(speed)}}{v} = \sqrt{2gy} \Rightarrow$

$$\frac{dV}{dt}^{\,\text{volume}} = -\underset{\text{ft}^2}{\frac{5}{3\cdot 12^2}}\,v = -\frac{5\sqrt{2g}}{3\cdot 12^2}\sqrt{y} = -\frac{40}{3\cdot 12^2}\sqrt{y}$$

($g = 32$ ft/sec²)

At the same time, $\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = 4 \cdot y'$, since $\frac{dV}{dy}$ = cross-sectional area.

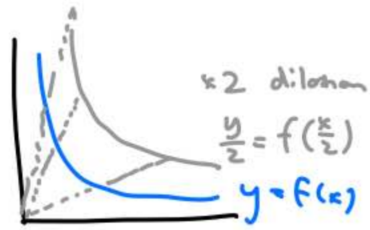
So we get the separable equation $y' = \frac{-10}{3\cdot 12^2}\sqrt{y} \Rightarrow \int \frac{dy}{\sqrt{y}} = -\int \frac{10}{3\cdot 12^2}\,dt$

$\Rightarrow 2\sqrt{y} = \frac{-10}{3\cdot 12^2}t + 2C \Rightarrow y = \left(-\frac{5}{3\cdot 12^2}t + C\right)^2, \quad C = \sqrt{2} \Rightarrow$

$$\left.\begin{aligned} &\text{when } y = 1, \quad t = \frac{12^2\cdot 3}{5}(\sqrt{2}-1) \sim 35.8 \text{ sec} \\ &\phantom{\text{when }} y = 0, \quad t = \frac{12^2\cdot 3}{5}\sqrt{2} \sim 122.2 \text{ sec.}\end{aligned}\right\}$$

this is assuming no friction. The book introduces a discharge coefficient = 0.6, and the effect is to divide these answers by 0.6, giving 59.6 sec. resp. 203.6 sec. //

Technique # 2 : $\boxed{\text{using homogeneity of the RHS}}$



Suppose we want to solve

$$y' = F(x,y)$$

where $F(tx, ty) = F(x,y)$ for all $t \neq 0$. Then solutions are all "dilations" of each other: if $y = f(x)$ solves it, i.e. $f'(x) = F(x, f(x))$, then consider $\frac{y}{k} = f(\frac{x}{k})$ (i.e. $y = k f\left(\frac{x}{k}\right)$): we have

$$y' = \frac{d}{dx} k f\left(\tfrac{x}{k}\right) = f'\left(\tfrac{x}{k}\right) = F\left(\tfrac{x}{k}, f\left(\tfrac{x}{k}\right)\right) = F\left(x, k f\left(\tfrac{x}{k}\right)\right) = F(x,y).$$

To find one solution, the trick is to substitute $v = y/x$ (i.e. $y = xv$), which yields

$$\begin{aligned} y' &= F(x,y) \\ x'v + xv' &= F(x, vx) \\ v + xv' &= F(1, v) \\ v' &= \frac{F(1,v) - v}{x} \end{aligned}$$

and then using Technique # 1

$$\int \frac{dv}{F(1,v) - v} = \int \frac{dx}{x}$$

Ex / Consider the two equations

(A) $y' = \frac{y^2 - x^2}{2xy}$ and (B) $y' = \frac{2xy}{x^2 - y^2}$.

The graph of a solution to a DE is called an integral curve.

Of course, at a point (x_0, y_0) on such a curve, y' is the slope. Since the product of $\frac{y^2-x^2}{2xy} \cdot \frac{2xy}{x^2-y^2} = -1$, the integral curves of these equations passing through any (x_0, y_0) are orthogonal (perpendicular).

For (A), $y = vx$ yields $v + xv' = \frac{x^2v^2 - x^2}{2x^2v} = \frac{v^2-1}{2v}$

$\rightsquigarrow xv' = \frac{-1-v^2}{2v} = -\frac{1}{2}\,\frac{v^2+1}{v} \rightsquigarrow -\int \frac{2v\,dv}{v^2+1} = \int \frac{dx}{x}$

$\rightsquigarrow \log x + C_0 = -\log(v^2+1) \rightsquigarrow C x^{-1} = (v^2+1)$

$\underset{\cdot x^2}{\rightsquigarrow} Cx = y^2 + x^2 \rightsquigarrow \left(x - \frac{C}{2}\right)^2 + y^2 = \frac{C^2}{4}$. These are circles of radius $\frac{C}{2}$ centered at $\left(\frac{C}{2}, 0\right)$ — i.e. centered on x-axis & passing thru the origin.

For (B), $y = vx$ yields $v + xv' = \frac{2x^2v}{x^2 - x^2v^2} = \frac{2v}{1-v^2}$

$\rightsquigarrow xv' = \frac{v+v^3}{1-v^2} \rightsquigarrow \int \frac{dx}{x} = \int \frac{(1-v^2)\,dv}{v(1+v^2)} = \int \left(\frac{1}{v} - \frac{2v}{1+v^2}\right) dv$

$= \log v - \log(1+v^2) = \log\left(\frac{v}{1+v^2}\right) \rightsquigarrow x = \frac{Cv}{1+v^2} = \frac{Cy/x}{1+y^2/x^2} = \frac{Cy}{x+y^2/x}$

$\rightsquigarrow x^2 + y^2 - Cy = 0$ gives circles passing thru the origin & centered on y-axis. So we have proved that these two families of circles are orthogonal wherever they meet!