Lecture 34 : Convergence of series

Ex/ Here are a few examples of convergent series and their sums:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \ , \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log(2) \ , \quad \sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{1}{3} \ , \quad \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \ . \quad //$$

<u>Convergence & Divergence tests</u>: (C = convergent, D = divergent)

(1) Basic divergence test: If $\sum a_n$ converges, then $\lim a_n = 0$.

(Hence if $\lim a_n \neq 0$ then $\sum a_n$ diverges.)

Proof: $S_n = \sum_{k=1}^{n} a_k \xrightarrow[\text{converges to}]{} S \implies a_n = S_n - S_{n-1} \to S - S = 0$. $\square$

Remark: Converse is false (e.g. consider $a_n = \frac{1}{n}$).

PROBLEM: $\sum_{n=1}^{\infty} \log\left(\frac{n^2+1}{2n^2+1}\right)$ C or D ?

(2) Geometric series: $\sum_{k=0}^{\infty} A z^k = \frac{A}{1-z}$ if $|z| < 1$, otherwise D.

Proof: See lect. 33 for $|z|<1$. If $|z| \geq 1$, then $a_k = Az^k \not\to 0$.
Apply (1). $\square$ $\square$

$$\text{Ex/} \quad \sum_{n=1}^{\infty} \frac{3^n - 2^{n+2}}{5^{n+1}} = \sum_{k=0}^{\infty} \frac{3}{5^2} \cdot \left(\frac{3}{5}\right)^k - \sum_{k=0}^{\infty} \frac{2^3}{5^2} \cdot \left(\frac{2}{5}\right)^k = \frac{3/5^2}{1-3/5} - \frac{2^3/5^2}{1-2/5}$$

$$= \frac{\cancel{5}}{2} \cdot \frac{3}{5\cancel{{}^2}} - \frac{\cancel{5}}{3} \cdot \frac{2^3}{5\cancel{{}^2}} = \frac{-7}{30} \ . \quad //$$

PROBLEM: (i) Compute $6 - 4 + \frac{8}{3} - \frac{16}{9} + \frac{32}{27} - \cdots$

(ii) Compute $\sum_{n=1}^{\infty} \left(\left(\frac{1}{6}\right)^n - \left(\frac{1}{6}\right)^{n+1}\right)$

(iii) For what values of x does $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n}$ converge?

Ex/ $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ can be integrated/differentiated to yield power series for other functions like $\frac{1}{(1-x)^m}$, $\frac{1}{1+x^2}$, $\log(1-x)$, $\arctan(x)$ valid for $|x|<1$. For example this gives $-\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$. (As we'll discuss in detail next week, one can justify this by bounding the

error in the Taylor approximation.) Substituting $x = -1$ gives
$\log(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$. //

(3) Telescoping Series: Given a sequence $\{b_k\}$, put $a_k := b_k - b_{k+1}$.
Then $\sum a_k$ C $\iff$ $\lim b_k$ exists.

Proof: $S_n = \sum_{k=1}^{n} a_k = (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_n - b_{n+1}) = b_1 - b_{n+1}$.
So $\lim b_n = B \iff \sum a_k = \lim_{n\to\infty} S_n = b_1 - B$. □

Ex/ $\sum_{k=1}^{\infty} \left(\sin\left(\frac{1}{k}\right) - \sin\left(\frac{1}{k+1}\right)\right) = \sin(1) - \lim_{n\to\infty} \sin\left(\frac{1}{n+1}\right) = \sin(1)$

[cannot write $\sum \sin\left(\frac{1}{k}\right) - \sum \sin\left(\frac{1}{k+1}\right)$: these diverge]

$$\sum_{k=1}^{\infty} \frac{1}{k^2+2k} = \sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$$
$$= \frac{1}{2} - \lim \frac{1/2}{k+1} + \frac{1}{4} - \lim \frac{1/2}{k+2} = 3/4. \quad //$$

[Warning: It's tempting to "telescope" where it doesn't apply. For example, in
$$\sum_{k=0}^{\infty} (-1)^k = \cancel{1 + (-1)} + \cancel{1 + (-1)} + \cancel{1 + (-1)} + \cdots = 0\ ?$$
$$= 1 + \cancel{(-1) + 1} + \cancel{(-1) + 1} + \cancel{(-1) + 1} + \cdots = 1\ ?$$
So $0 = 1$?

both conclusions are wrong. Recall that the sum of the series is the limit of $S_n := \sum_{k=0}^{n} (-1)^k$, which alternates between 0 & 1, so has no limit. That is, $\sum (-1)^k$ diverges.]

(4) Monotonic Series test: If $a_k \geq 0$ & $\sum_{k=1}^{n} a_k \leq B$ $(\forall n)$, then $\sum a_k$ converges. [Immediate consequence of Monotonic Sequence Thm.]

(5) Comparison test: If $a_k, b_k \geq 0$ & $a_k \leq c\, b_k$ for some $c \in \mathbb{R}^+$.
Then $\sum b_k$ C $\Rightarrow$ $\sum a_k$ C (i.e. $\sum a_k$ D $\Rightarrow$ $\sum b_k$ D).

Proof: Say $B = \sum b_k = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} b_k}_{\text{increasing sequence, bounded above by } B}$. Then $\underbrace{\sum_{k=1}^{n} a_k}_{\text{also an increasing sequence}} \leq c \sum_{k=1}^{n} b_k \leq cB$, apply (4). □

Ex/ $\underset{C}{\sum \frac{1}{2+3^n}} \leq \underset{C}{\sum \frac{1}{3^n}}$. //

$C \Longleftarrow C$

(6) Limit comparison test: Suppose $a_k, b_k > 0$ and $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$.

Then $\sum a_k$ & $\sum b_k$ converge or diverge together.

Proof: $\exists N$ s.t. $k \geq N \Rightarrow \frac{1}{2} < \frac{a_k}{b_k} < \frac{3}{2} \Longrightarrow$

$b_k \leq 2a_k$ & $a_k \leq \frac{3}{2} b_k$. Apply (5). $\square$

Remark: If $\lim \frac{a_n}{b_n} = 0$, then $\sum b_n$ C $\Rightarrow \sum a_n$ C. (Again, use (5).)

Ex/ $\sum \frac{n+3}{2^n(n+1)}$ converges since $\sum \frac{1}{2^n}$ does and

$$\lim_{n\to\infty} \frac{n+3/(2^n(n+1))}{1/2^n} = \lim \frac{n+3}{n+1} = 1. \quad //$$

(7) Integral test: If $a_n = f(n)$ for $f:[1,\infty) \to \mathbb{R}_{\geq 0}$ continuous and decreasing, then $\int_1^\infty f(x)\,dx$ and $\sum a_n$ C or D together.

[Remark: (i) We define $\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx$ for f continuous on $[a,\infty)$ and $\int_{a^+}^b f(x)\,dx := \lim_{c\to a^+} \int_c^b f(x)\,dx$ for f cts. on $(a,b]$. Skim over comparison tests for these in the book.

(ii) It's fine to use this test starting at some later term in the sequence, e.g. comparing $\sum_{k=m}^\infty a_k$ & $\int_m^\infty f(x)\,dx$.]

Proof: Write $b_n = \int_n^{n+1} f(x)\,dx$, so that $\int_1^\infty f(x)\,dx = \lim_{x\to\infty} \int_1^x f(x)\,dx$

$= \lim_{n\to\infty} \int_1^{n+1} f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^n b_k = \sum_{k=1}^\infty b_k$. Since f is decreasing, we have $a_n = f(n) \geq b_n \geq f(n+1) = a_{n+1}$. Apply (5) to

" $\sum_{n=1}^\infty a_n \geq \sum_{n=1}^\infty b_n$ " and " $\sum_{n=1}^\infty a_{n+1} \leq \sum_{n=1}^\infty b_n$ ". $\square$

Ex/ $\underbrace{\sum_{n=1}^\infty \frac{1}{n^s}}_{}= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \begin{cases} C & \text{for } s > 1 \\ D & \text{for } s \leq 1, \end{cases}$

this is called the "Riemann zeta function" when interpreted as a function of s

since $\int_1^\infty \frac{dx}{x^s} = \lim_{b\to\infty} \int_1^b x^{-s}\,dx = \lim_{b\to\infty} \left.\frac{x^{-s+1}}{-s+1}\right|_1^b$

$= \lim_{b\to\infty} \left(\frac{1}{s-1} - \frac{b^{1-s}}{s-1}\right) = \begin{cases} \frac{1}{s-1} & \text{if } s > 1 \\ \infty & \text{if } s \leq 1. \end{cases}$ //

PROBLEM: Which of $\sum_{n=2}^\infty \frac{1}{n(\log n)^2}$, $\sum_{n=1}^\infty \frac{\log n}{n^2}$, $\sum_{n=2}^\infty \frac{1}{n \log n}$, $\sum_{n=1}^\infty \frac{1}{n^{\sqrt{2}}}$, and $\sum_{n=2}^\infty \frac{1}{\sqrt{n} - 1}$ diverge?

Notes on this last one:

- $\int \frac{dx}{x(\log x)^2} = \frac{1}{\log x} \to 0 \xrightarrow[⑦]{} C$
- $\frac{(\log n)/n^2}{1/n^{3/2}} = \frac{\log n}{\sqrt{n}} \to 0 \xrightarrow[⑥]{} C$
- $\int \frac{dx}{x \log x} = \log(\log(x)) \to \infty \xrightarrow[⑦]{} D$
- C by Ex
- D by ⑥