Lecture 35 : More convergence tests

Before we move on from the integral test, here is an interesting example of improper integrals.

Ex/ $\Gamma(s) := \int_{0^+}^{\infty} t^{s-1} e^{-t}\, dt \quad (s > 0)$ is called the Gamma function.

[The integral converges: the $\int_{0^+}^{1} = \lim_{a \to 0^+} \int_a^1$ part, by comparison with $\lim_{a \to 0^+} \int_a^1 t^{s-1} dt = \lim_{a \to 0^+} \frac{t^s}{s}\Big|_a^1 = \frac{1}{s}$; the $\int_1^{\infty} = \lim_{b \to \infty} \int_1^b$ part, b/c $t^{s-1} e^{-t}$ is eventually smaller than t^{-2} (as $\frac{t^{s-1} e^{-t}}{t^{-2}} = \frac{t^{s+1}}{e^t} \to 0$) and $\int_1^{\infty} t^{-2} dt$ converges.]

It has the really cool property that

$$\Gamma(s+1) = \int_{0^+}^{\infty} t^s e^{-t} dt \underset{\uparrow}{=} -\frac{t^s}{e^t}\Big|_{0^+}^{\infty} + s\int_{0^+}^{\infty} t^{s-1} e^{-t} dt = 0 + s\Gamma(s)$$

$$\left(\begin{array}{cc} \multicolumn{2}{c}{\int \text{ by parts :}} \\ u = t^s & dv = e^{-t} dt \\ du = s t^{s-1} ds & v = -e^{-t} \end{array}\right) \qquad = s\,\Gamma(s).$$

Since $\Gamma(1) = \int_{0^+}^{\infty} e^{-t} dt = -e^{-t}\Big|_{0^+}^{\infty} = 1 \ (= 0!)$, $\Gamma(n+1) = n!$ //

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⑧ Ratio test: If $a_n > 0$ and $\rho := \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$,

- $0 \le \rho < 1 \Rightarrow \sum a_n$ converges
- $\rho = 1$: inconclusive (e.g., $a_n = \frac{1}{n}$ vs. $\frac{1}{n^2}$)
- $\rho > 1$ (or ∞) $\Rightarrow \sum a_n$ diverges.

Ex/ $\sum \frac{2^n}{n!}$ has $\rho = \lim \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim \frac{2}{n+1} = 0 \Rightarrow C$

$\sum \frac{n!}{n^n}$ has $\rho = \lim \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim \frac{n^n}{(n+1)^n} = \frac{1}{\lim(1+\frac{1}{n})^n} = \frac{1}{e} \Rightarrow C$ //

Proof: If $\rho < 1$, then taking $r \in (\rho, 1)$ there exists $N \in \mathbb{P}$ s.t.

$n \geq N \Rightarrow \frac{a_{n+1}}{a_n} < r \Rightarrow a_{N+k} < r a_{N+k-1} < \cdots < r^k a_N$

$\Rightarrow a_n \leq \frac{a_N}{r^N} r^n = C \cdot r^n \Rightarrow \sum a_n$ converges by comparison to (convergent) geom. series $\sum r^n$.

If $\rho > 1$, then for $n \geq N$, a_n is increasing $\Rightarrow a_n$ doesn't limit to $0 \Rightarrow \sum a_n$ diverges by basic divergence test. □

(9) Root test: If $a_n > 0$ and $\varrho := \lim_{n\to\infty} \sqrt[n]{a_n}$,

- $0 \leq \varrho < 1 \Rightarrow \sum a_n$ converges
- $\varrho = 1$: inconclusive
- $\varrho > 1$ (or ∞) $\Rightarrow \sum a_n$ diverges.

Ex/ $\sum \frac{1}{(\log n)^n}$ has $\varrho = \lim \frac{1}{\log n} = 0 \Rightarrow C$

$\sum \frac{n!}{n^n}$ has $\varrho = \lim \frac{\sqrt[n]{n!}}{n} = \frac{1}{e} \Rightarrow C$

$$\sum_{k=1}^{n-1} \log(k) \leq \int_1^n \log(x)\,dx \leq \sum_{k=2}^{n} \log(k) \quad \text{(draw the picture!)}$$

$$\exp\left(\ \log((n-1)!) \leq n \log n - n \leq \log(n!)\right.$$

$$(n-1)! \leq \frac{n^n}{e^n} \leq n! \leq \frac{n^{n+1}}{e^n}$$

$\sqrt[n]{\ }$

$$\frac{n}{e} \leq \sqrt[n]{n!} \leq \frac{n}{e} \cdot n^{1/n}$$

$/n$

$$\frac{1}{e} \leq \frac{\sqrt[n]{n!}}{n} \leq \frac{1}{e} n^{1/n}$$

$\to 1/e$ by squeeze lemma //

Proof: If $\varrho < 1$, take $r \in (\varrho, 1)$. There exists N s.t. $n \geq N \Rightarrow a_n^{1/n} < r$ $\Rightarrow a_n < r^n$. Since $\sum r^n$ C, done by comp. test. The $\varrho > 1$ case is as in the last proof. □

(10) <u>Alternating series test</u>: A series of the form $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges if b_n is decreasing with limit 0.

<u>Proof</u>: $S_{2k} = S_{2k-2} + \underbrace{b_{2k-1} - b_{2k}}_{>0}$ is increasing

$S_{2k-1} = S_{2k-3} \underbrace{- b_{2k-2} + b_{2k-1}}_{<0}$ is decreasing

N.B. Here $S_n = \sum_{j=1}^{n} b_j$ are the partial sums as usual.

$$\Rightarrow S_2 < S_{2k} \left(= S_{2k-1} - b_{2k}\right) < S_{2k-1} < S_1 .$$

bounded below by

bounded above by

By monotonic sequence Theorem, $S_{2k} \to S'$ and $S_{2k-1} \to S''$ have limits; and $S' - S'' = \lim S_{2k} - \lim S_{2k-1}$
$= \lim (S_{2k} - S_{2k-1}) = \lim(-b_{2k}) = 0$. So $S_n \to S' = S''$, and the series converges. $\square$

Ex/ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent by (10).

What is its sum? Write

$$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$$
$$- \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$
$$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \frac{1}{n}\left(\frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n}\right)$$
$$= \frac{1}{n}\left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right) = \underbrace{\frac{1}{n}\sum_{i=1}^{n} f(x_i)}_{\text{Riemann sum for } \int_1^2 f(x)\,dx} \qquad f(x) = \frac{1}{x}$$
$$\xrightarrow[n\to\infty]{} \int_1^2 \frac{dx}{x} = \log(2).$$

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<u>Problem</u>: Which of $\sum (-1)^{n+1} \frac{n}{n^2+1}$, $\sum (-1)^n \frac{3n+5}{n+1}$,
$1 - \frac{1}{4} + \frac{1}{3} - \frac{1}{16} + \frac{1}{5} - \frac{1}{36} + \frac{1}{7} - \frac{1}{64} + \frac{1}{9} - \cdots$, and $\sum (-1)^{n+1} \frac{n^2}{2^n}$ converge?

Answers:

- C
- D — $b_k \not\to 0$
- D — $b_k \to 0$ but not decreasing (can prove D)
- C — $b_k \to 0$ and $\downarrow$ after 3rd term!