Lecture 36: Rearrangements

these could be in $\mathbb{C}$

Definition: We say that $\sum a_n$ is absolutely convergent (AC) if $\sum |a_n|$ converges.

Proposition: If $\sum a_n$ is AC, then $\sum a_n$ converges.

Proof: Assume $a_n \in \mathbb{R}$ $(\forall n)$, and define $b_n := a_n + |a_n|$. Since $0 \le b_n \le 2|a_n|$, and $\sum 2|a_n|$ converges, $\sum b_n$ converges by the comparison test. Hence $\sum a_n = \sum b_n - \sum |a_n|$ converges.

(If $a_n = u_n + i v_n \in \mathbb{C}$ ($u_n, v_n \in \mathbb{R}$), then $|u_n|, |v_n| \le |a_n| \Rightarrow \sum u_n, \sum v_n$ C $\Rightarrow \sum a_n = \sum u_n + i \sum v_n$ C.) $\square$

Definition: $\sum a_n$ is conditionally convergent (CC) if it is C but not AC.

Ex/ $\sum (-1)^n \frac{1}{n}$ is CC. //

Definition: A permutation of $\mathbb{P}$ is a 1-1/onto function $\sigma : \mathbb{P} \to \mathbb{P}$.

A rearrangement of a sequence $\{a_n\}$ (& the corresponding series) is a sequence of the form $b_n := a_{\sigma(n)}$, for some permutation σ.

Riemann's Theorem on Rearrangements of Series ($a_n \in \mathbb{R}$)

(i) $\sum a_n$ AC $\Longrightarrow$ every rearrangement is convergent with the same sum

(ii) $\sum a_n$ CC $\Longrightarrow$ for any $r \in \mathbb{R}$, there exists a rearrangement $\sum b_n$ of $\sum a_n$ with sum r.

Proof: Write

- $\{b_k\}$ for the rearrangement, and
- $B_n = \sum_{k=1}^{n} b_k$ and $A_n = \sum_{k=1}^{n} a_k$

(i) Also write $A_n^* = \sum_{k=1}^{n} |a_k|$, $S^* = \sum_{k=1}^{\infty} |a_k|$, $S = \sum_{k=1}^{\infty} a_k$.

The $\{B_n\}$ are bounded above by S^* (why?), while the partial sums $A_n \to S$ and $A_n^* \to S^*$. Hence for any given $\epsilon > 0$, $\exists\ N$ s.t. $|A_n - S| < \frac{\epsilon}{2}$ & $|A_n^* - S^*| < \frac{\epsilon}{2}$.

Let M be such that $\{\sigma^{-1}(1), \ldots, \sigma^{-1}(N)\} \subset \{1, \ldots, M\}$, hence $\{1, \ldots, N\} \subset \{\sigma(1), \ldots, \sigma(M)\}$. Then for $n \geq M$,

$$|B_n - S| = |B_n - A_N + A_N - S| \leq \underbrace{|A_N - S|}_{< \epsilon/2} + |B_n - A_N| < \epsilon/2 + \epsilon/2 = \epsilon,$$

$$|B_n - A_N| = \left| \sum_{k=1}^{n} a_{\sigma(k)} - \sum_{k=1}^{N} a_k \right|$$

(includes all of these)

$$\leq \sum_{k=N+1}^{\infty} |a_k|$$

$$= S^* - A_N^* < \frac{\epsilon}{2}$$

which completes the proof that $B_n \to S$.

(ii) Write $a_n^+ := \begin{cases} a_n, & a_n > 0 \\ 0, & a_n \leq 0 \end{cases}$, $a_n^- = \begin{cases} a_n, & a_n < 0 \\ 0, & a_n \geq 0 \end{cases}$.

Since $\frac{1}{2}|a_n|$ diverges, $\sum a_n^+ = \frac{1}{2}\sum a_n + \frac{1}{2}\sum |a_n|$ and $\sum a_n^- = \frac{1}{2}\sum a_n - \frac{1}{2}\sum |a_n|$ (C, D) } both diverge

— as they are series of positive terms, to infinity.

Take $b_1, \ldots, b_{N_1}$ to be just enough (i.e. $b_1 + \cdots + b_{N_1 - 1} \leq r$) nonzero terms from a_k^+ (preserving the order of these terms) s.t. $B_{N_1} > r$. This is possible since $\sum a_k^+ = \infty$.

Take $b_{N_1+1}, \ldots, b_{N_2}$ to be just enough terms from a_k^- (preserving the order of these terms too) s.t. $B_{N_2} < r$.

Continuing in this fashion, we get $B_{N_3} > r$, $B_{N_4} < r$, etc., where $|B_{N_j} - r| \leq |b_{N_j}|$. But since $a_k^+, a_k^- \longrightarrow 0$ (due to convergence of $\sum a_n$), $|b_{N_j}| \to 0 \implies |B_{N_j} - r| \to 0$ $\implies |B_n - r| \to 0$. So $\sum b_n = r$. $\square$

Ex/ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots = \log(2)$.

Consider the rearrangement

$$\underbrace{1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6}} + \cdots$$

← this is S_9 $(m=3)$

with partial sums

$$S_{3m} = \sum_{k=1}^{2m} \frac{1}{2k-1} - \sum_{k=1}^{m} \frac{1}{2k} = \left(\sum_{k=1}^{4m} \frac{1}{k} - \sum_{k=1}^{2m} \frac{1}{2k} \right) - \sum_{k=1}^{m} \frac{1}{2k}$$

$$= \mathcal{S}_{4m} - \frac{1}{2}\mathcal{S}_{2m} - \frac{1}{2}\mathcal{S}_m, \quad \text{where } \mathcal{S}_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

To evaluate this, consider the decreasing sequence a_n $(\to 0)$ with $a_{2m-1} = \frac{1}{m}$, $a_{2m} = -\int_m^{m+1} \frac{dx}{x}$. By the alternating series test, $\sum a_n$ converges with sum the <u>Euler-Maclaurin constant</u>

$$\gamma := \lim_{m\to\infty} \sum_{k=1}^{2m-1} a_k = \lim_{m\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{m} - \log m \right).$$

From this we obtain $\mathcal{S}_m = \log m + \gamma + o(1)$, hence

$$S_{3m} = \left(\log 4m + \cancel{\gamma} + o(1)\right) - \left(\tfrac{1}{2}\log 2m + \cancel{\tfrac{\gamma}{2}} + o(1)\right) - \left(\tfrac{1}{2}\log m + \cancel{\tfrac{\gamma}{2}} + o(1)\right)$$

$$= \cancel{\log m} - \cancel{\tfrac{1}{2}\log m} - \cancel{\tfrac{1}{2}\log m} + \log 4 - \tfrac{1}{2}\log 2 + o(1)$$

$$= \tfrac{3}{2}\log 2 + o(1) \longrightarrow \tfrac{3}{2}\log(2). \quad //$$

Finally, have a look at the Abel & Dirichlet tests in the text. I won't cover those in class for now.

Appendix: Here's a pocket guide I used in Calc 2 —

Given a series, how do you know which test to use?

[1] Is it a geometric series, or p-series? (Then done.) ← i.e. $\sum \frac{1}{n^p}$

[2] Does it have $\lim_{n\to\infty} a_n \neq 0$? (If so, D.)

[3] Is it a positive term series? If so, use one of:

- [a] — Basic C.T.
- [b] — Limit C.T. (esp. if $a_n = \frac{P(n)}{Q(n)}$ ← polynomials)
- [c] — Ratio test (esp. if there is an $n!$, or a c^n)
- [d] — Root test (esp. if have n^n or $P(n)^n$)
- [e] — Integral test (if $a_n = f(n) \to 0$ with $\int f\,dx$ integrable)

[4] Does it have both positive and negative terms?

- [a] — If $a_n = (-1)^n b_n$, with b_n decreasing to 0, then C.
- [b] — otherwise, use [3] applied to $|a_n|$

[5] Is it a power series?

- Use ratio/root test to find the convergence interval
- then use [3] & [4] to check for convergence at the endpoints of the interval.

Maybe this is helpful, & maybe just redundant.