Lecture 37: Sequences & series of functions

Definition: (i) Let S be a subset of $\mathbb{R}$ or $\mathbb{C}$. A sequence of functions $\{f_n\}$ is a map from $\mathbb{P} \times S \longrightarrow \mathbb{R}$ or $\mathbb{C}$.
$(n, z) \longmapsto f_n(z)$

It converges (pointwise) on S to f if $\lim_{n\to\infty} f_n(z) = f(z)$ $\forall z \in S$, in which case we write $f_n \longrightarrow f$.

(ii) A series of functions $\sum_{k=1}^{\infty} f_k$ — i.e. the sequence of n^{th} partial sums $g_n := \sum_{k=1}^{n} f_k$ — converges on S with sum f if $g_n \to f$. It is absolutely convergent (AC) on S if $\sum |f_k|$ converges.

Ex 1/ Let $S := [0,1]$, $f_n(x) := nx(1-x^2)^n$. Do we have
$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$? Clearly this is desirable, e.g. if we want to be able to integrate function series termwise.
But it fails here: $\lim_{n\to\infty} f_n(x) =: f(x)$ is ZERO on S, while
$\lim_{n\to\infty} \int_0^1 nx(1-x^2)^n\,dx = \frac{1}{2}$. //

Definition: (i) $\{f_n\}$ is uniformly convergent (UC) on S (to f) if
$\forall \epsilon > 0$ $\exists N \in \mathbb{P}$ s.t. $n \geq N \implies |f_n(z) - f(z)| < \epsilon$ for every $z \in S$.

(ii) $\sum f_n$ is UC if $g_n = \sum_{k=1}^{n} f_k$ is.

Remark: The sequence in Ex. 1 is not UC, because no matter how large n is, we still have a point $x_n = \frac{1}{n}$ at which
$|f_n(x_n) - f(x_n)| = \left| n \cdot \frac{1}{n}\left(1 - \left(\tfrac{1}{n}\right)^2\right)^n - 0 \right| = \left|1 - \frac{1}{n^2}\right|^n > \frac{1}{2}$.

Theorem 1: Given $f_n \to f$ UC on $\mathcal{S} \subset \mathbb{R}$:

(a) f_n all continuous $\Longrightarrow$ f continuous

(b) $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$ for $[a,b] \subseteq \mathcal{S}$.

Proof: (a) Given $x_0 \in \mathcal{S}$ and $\epsilon > 0$, $\exists\, N \in \mathbb{P}$ s.t. $|f_N(x) - f(x)| < \epsilon/3$ $\forall x \in \mathcal{S}$, and $\delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \epsilon/3$.

So $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| \le |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|$ $< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$, done.

$\Rightarrow$ so is $|f_n - f|$

(b) First, since f is cts., it is integrable. Now given $\epsilon > 0$, $\exists\, N \in \mathbb{P}$ s.t. $n \ge N \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ $\forall x \in [a,b] \Longrightarrow$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b (f_n(x) - f(x))\,dx \right| \le \int_a^b |f_n(x) - f(x)|\,dx < \epsilon. \qquad \square$$

Corollary: If $\sum f_n$ converges uniformly to f, then $\sum \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

Remark: By varying b, this says that a primitive of f may be obtained by integrating $\int_a^x$ termwise.

So if $\sum f_n = f$ converges on $\mathcal{S}$, how do we ensure/check UC?

Theorem 2: If (for each n) $|f_n| \le M_n$ ($\in \mathbb{R}_{\ge 0}$) on all of $\mathcal{S}$, and $\sum M_n$ converges, then $\sum f_n$ is AC & UC on $\mathcal{S}$. (Here we can have $\mathcal{S} \subset \mathbb{C}$.)

Proof: AC is clear from $|f_n(x)| \le M_n$ by the basic comparison test.

Now $\sum M_n$ C $\Longrightarrow$ $\forall \epsilon > 0\ \exists\, N \in \mathbb{P}$ s.t. $\sum_{k=n+1}^{\infty} M_k < \epsilon$ for $n \ge N$,

and so $\left| f(x) - \sum_{k=1}^{n} f_k(x) \right| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \le \sum_{k=n+1}^{\infty} |f_k(x)| \le \sum_{k=n+1}^{\infty} M_k < \epsilon$ $\forall x \in \mathcal{S}$. $\square$

Corollary: (a) For any power series $\sum a_n z^n$, there exists a radius of convergence $r \in \mathbb{R}_{\ge 0} \cup \{\infty\}$ such that the series AC on $|z| < r$ and diverges for any $z_0 \in \mathbb{C}$ with $|z_0| > r$.

(b) If $R \in (0,r)$ then $\sum a_n z^n$ UC on $|z| \leq R$.

Proof: Every power series converges at $z=0$. If $\sum a_n z^n$ converges at no other point, then the statements hold with $r=0$.

So suppose $\sum a_n z_0^n$ converges for some $z_0 \neq 0$. Then $a_n z_0^n \to 0$ $\Rightarrow |a_n z_0^n| < 1$ for $n \geq N$. Given $R \in (0, |z_0|)$, $|z| \leq R \Rightarrow$

$$|a_n z^n| = |a_n z_0^n| \left|\frac{z}{z_0}\right|^n \leq \left(\frac{R}{|z_0|}\right)^n =: M_n \underset{\text{Thm. 2}}{\Longrightarrow} \sum a_n z^n \text{ AC/UC on } |z| \leq R.$$

Moreover, we have AC on $|z| < |z_0|$ by varying R.

Let $\mathcal{A} := \{ \rho \in \mathbb{R}^+ \mid \sum a_n z^n \text{ converges for some } z \text{ with } |z| = \rho \}$. This is nonempty by hypothesis; if it is unbounded, then the statements of the Corollary hold with $r = \infty$ (by the last paragraph). Finally, if $\mathcal{A}$ has an upper bound, let $r := \sup \mathcal{A}$. Given any $R < r$ there exists $z_0 \in \mathbb{C}$ with $\sum a_n z_0$ convergent & $R < |z_0| < r$, and so we are done by the last paragraph. $\square$

In practice, one uses the root/ratio tests to <u>calculate</u> the radius of convergence of a given power series.

Ex 2/ What are the radii of convergence in each of the following cases?

$$\sum n^n z^n, \quad \sum \frac{z^n}{(n+1)2^n}, \quad \sum \frac{z^n}{n!}, \quad \sum \binom{2n}{n}^2 z^n. \quad //$$