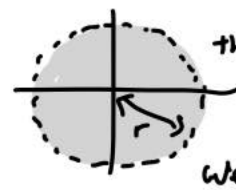
Lecture 38 : Power series

Last time we introduced the notion of uniform convergence (UC) of a function series $\sum_{n=0}^{\infty} f_n(x)$ on a set S. This means that for each $\epsilon > 0$ we can find an N s.t. $n \geq N \Rightarrow \left| \sum_{k=n+1}^{\infty} f_k(x) \right| < \epsilon$ on S.
It guarantees that the sum is continuous if the $\{f_n\}$ are, and that it may be integrated termwise. We showed that for a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists a unique $r \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ (the radius of convergence) so that:



the "disk of convergence" $|z| < r$ in $\mathbb{C}$. We can't say anything in general about convergence/divergence on the boundary circle.

in $\mathbb{R}$, this means $-r < z < r$

- the series is AC on $|z| < r$
- the series is UC on $|z| \leq R$, for any $R < r$
- the series diverges on $|z| > r$. $z > r$ or $z < -r$

We can also replace z everywhere by $z - a$; then the disk is centered at a instead of 0. Some immediate consequences:

- $f(x) := \sum_{n=0}^{\infty} a_n (z-a)^n$ is continuous on $(a-r, a+r)$
- $\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$ on $(a-r, a+r)$ [$\Rightarrow$ radius of conv. of this is at least r]

Proposition: Integration and differentiation termwise preserve the radius of convergence. Moreover, $f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$.

Proof: Given $a < x < y < a+r$, the MVT yields $u_n \in (x,y)$ (for each n) s.t. $\frac{(y-a)^n - (x-a)^n}{y-x} = n(u_n - a)^{n-1}$. Since $\sum a_n (x-a)^n$, $\sum a_n (y-a)^n$ AC,

$$\frac{f(y) - f(x)}{y - x} = \sum_{n=0}^{\infty} a_n \frac{(y-a)^n - (x-a)^n}{y-x} = \sum_{n=0}^{\infty} n a_n (u_n - a)^{n-1} \quad \text{is AC,}$$

and so by comparison $\left(|x-a|^{n-1} \leq |u_n - a|^{n-1} \right)$ $\underbrace{\sum_{n=1}^{\infty} n a_n (x-a)^{n-1}}_{=: g(x)}$ is AC.

Since x can be taken arbitrarily close to $a+r$, the radius of convergence of $\sum n a_n (x-a)^{n-1}$ is at least r. So neither integration nor differentiation of series decreases radius of convergence, and $\therefore$ neither can change it. Moreover, $\int_a^x g(x)\,dx = \sum_{n=1}^{\infty} \int_a^x n a_n (x-a)^{n-1}\,dx$ $= \sum_{n=1}^{\infty} a_n (x-a)^n = f(x) - a_0 \Rightarrow f'(x) = g(x)$. □

Ex/ Since the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ works on $|x|<1$, integrals & derivatives work there too: hence $-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ holds there, as well as the series for $\arctan(x)$ (by integrating $\frac{1}{1+x^2} = \sum (-1)^n x^{2n}$). //

Corollary: If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ on $(a-r, a+r)$, then f is infinitely differentiable there, and $a_k = \frac{f^{(k)}(a)}{k!}$ for each k.

Proof: Apply the Proposition k times and plug in $x=a$. □

Conversely, if f is $\overbrace{\text{infinitely differentiable}}^{\text{"smooth"}}$ at a, then we can form the Taylor series of f at a: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. If this equals f in some neighborhood of a, then f is called an analytic function there. Unfortunately, not all smooth functions are analytic:

Ex/ $$f(x) := \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$ has $f^{(n)}(0) = 0$ $(\forall n)$,

so its Taylor series is 0. Obviously f isn't 0.

(See your HW.) //

need not be the radius of convergence of anything

Theorem: If f is smooth on $I = (a+R, a-R)$, and $|f^{(n)}(x)| \le B_n$ $(\forall x \in I)$ for some sequence $\{B_n\}$ with $\frac{r^n B_n}{n!} \to 0$ $\forall r \in (0,R)$, then the Taylor sequence of f at a converges to f on I:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \forall x \in I.$$

Proof: We need to show $E_n(x) := f(x) - T_n(x) \to 0$ for $x \in I$.

Since $E_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt$, writing $R := |x-a|$ we have

$$|E_n(x)| \leq \frac{B_{n+1}}{n!}\int_a^x (x-t)^n \, dt = \frac{B_{n+1} R^{n+1}}{(n+1)!} \longrightarrow 0. \qquad \square$$

Ex/ $|\sin^{(n)}(x)| \leq 1 \quad \forall x \in \mathbb{R}$, and $\frac{R^n}{n!} \to 0$ for any R,

so Thm. $\Rightarrow \sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$. Same argument for cosine

$\Rightarrow \cos(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}$. For $f(x) = e^x$, $f^{(n)}(x) = e^x$ on

$(-R, R)$ is bounded by $B_n = e^R$, and $\frac{R^n e^R}{n!} \to 0$: so Thm. $\Rightarrow$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. All of these hold on any $(-R, R)$, so hold

on $\mathbb{R}$ — we already know these series had ∞ radius of convergence. //

Ex/ $f(x) = (1+x)^\alpha$, $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ (i.e. α not an integer).

We know that $\frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \binom{\alpha}{n}$, so

the series is $\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$. By the ratio test $\left(\rho = \lim_{n\to\infty} \frac{\left|\binom{\alpha}{n+1} x^{n+1}\right|}{\left|\binom{\alpha}{n} x^n\right|} \right.$

$\left. = \lim_{n\to\infty} \frac{|\alpha - n|}{n+1}|x| = |x| \right)$, it is AC for $|x| < 1 \implies$

$\binom{\alpha}{n} x^n \to 0$ for $|x| < 1$. $\left(\Rightarrow n\binom{\alpha}{n} x^n \text{ also} \to 0 \right)$.

Now going back to the proof above, we get

(with $a = 0$, $|x| < R < 1$) $\quad |E_{n-1}(x)| = \left| \int_0^x n\binom{\alpha}{n}(1+t)^{\alpha-n}(x-t)^{n-1} \, dt \right|$

$$\leq n\left|\binom{\alpha}{n}\right| \int_0^x |1+t|^{\alpha-1} \underbrace{\left|\frac{x-t}{1+t}\right|^{n-1}}_{\text{maximum value of } |x| \text{ (at } t=0)} dt \leq n\left|\binom{\alpha}{n}\right| |x|^{n-1} \int_0^x |1+t|^{\alpha-1} \, dt$$

(or $\int_x^0$ if $x<0$)

$$\leq C\, n\left|\binom{\alpha}{n}\right| |x|^n \longrightarrow 0.$$

So $f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ on any $(-R, R)$ with $R < 1$, hence on $(-1, 1)$. //

Application to differential equations

How to solve $x(1-x)y'' + (1-2x)y' - \frac{2}{9}y = 0$ in a neighborhood of 0? Try formally substituting $y = f(x) := \sum a_n x^n$: we get (for all $x \in$ some interval of convergence)

$$0 = x(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + (1-2x)\sum_{n=1}^{\infty} n a_n x^{n-1} - \frac{2}{9}\sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} \frac{2}{9} a_n x^n$$

reindex
$$= \sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} \frac{2}{9} a_n x^n$$

$$= \left(a_1 - \frac{2}{9}a_0\right) + \left(4a_2 - \frac{20}{9}a_1\right)x + \sum_{n=2}^{\infty}\left(\underbrace{(n^2+2n+1)}_{(n+1)^2} a_{n+1} - \underbrace{(n^2+n+\tfrac{2}{9})}_{(n+\frac{1}{3})(n+\frac{2}{3})} a_n\right)x^n.$$

Taking $a_0 = 1$, we get $a_1 = \frac{2}{9}$, $a_2 = \frac{10}{81}$, $a_{n+1} = \frac{(n+\frac{1}{3})(n+\frac{2}{3})}{(n+1)^2} a_n$

$$\Rightarrow a_n = \frac{\left(\frac{1}{3}\cdot\frac{4}{3}\cdot\ldots\cdot\frac{3n-2}{3}\right)\left(\frac{2}{3}\cdot\frac{5}{3}\cdot\ldots\cdot\frac{3n-1}{3}\right)}{(n!)^2} = \frac{\left(\frac{1}{3}\cdots\frac{3n-2}{3}\right)\left(\frac{2}{3}\cdots\frac{3n-1}{3}\right)\left(\frac{3}{3}\cdots\frac{3n}{3}\right)}{(n!)^3}$$

$$= \frac{(3n)!}{(n!)^3 3^{3n}},$$ and so

$$f(x) = \sum_{n=0}^{\infty} \frac{3n!}{n!^3}\left(\frac{x}{27}\right)^n$$ which has radius of convergence

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{(n+1)^3\, 3^3}{(3n+3)(3n+2)(3n+1)} = \lim_{n\to\infty} \frac{(n+1)^3}{(n+1)(n+\frac{2}{3})(n+\frac{1}{3})} = 1.$$

This is an example of a "hypergeometric function", a new type of function discovered by Gauss that can't be written in terms of the ones you already know.