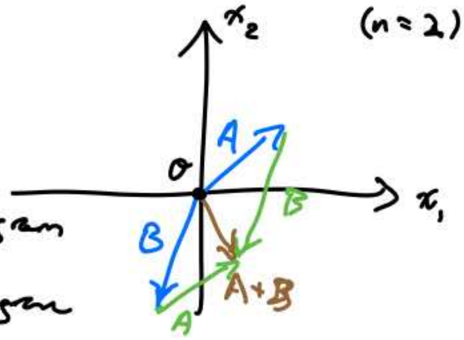
Lecture 39: Vector spaces

In order to use Calculus to further study differential equations, to treat curvilinear motion (e.g. Kepler's Laws), and ultimately to study functions of several variables, we will need to be comfortable working with vectors and matrices. Today we start the vector part, but without a layer of abstraction that will come later: for now we'll study vectors as ordered n-tuples of real numbers

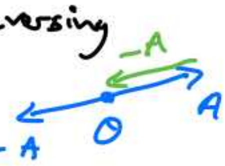
$$A = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$$

with the slight change that we shall write "V_n" for $\mathbb{R}^n$.
("Ordered" means that A equals $B = (b_1, \dots, b_n) \iff a_i = b_i \ (\forall i)$.)
The following algebraic operations are defined:

- vector addition : $A + B := (a_1 + b_1, \dots, a_n + b_n)$
  - may be visualized by "adding arrows head to tail"
    geometric vectors
  - commutativity $(A + B = B + A)$
    corresponds to the fact that the vectors A, B give the sides of a parallelogram (or that opposite vertices of any parallelogram have the same sum)
    follows from the same property for real numbers



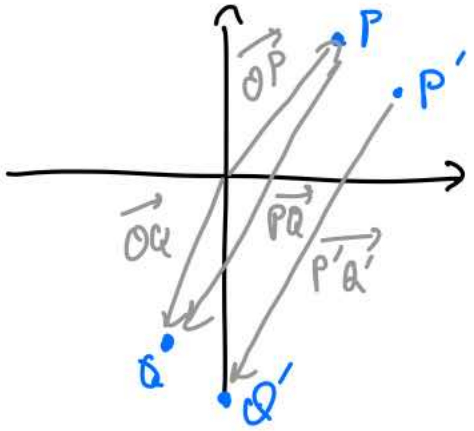
  - associativity $(A + (B + C) = (A + B) + C)$
  - $O + A = A$, where $O := (0, 0, \dots, 0)$
- scalar multiplication : $cA = (ca_1, ca_2, \dots, ca_n)$ for $c \in \mathbb{R}$
  - may be visualized by dilation of arrows (and reversing their direction if $c < 0$), e.g. $-A + A = O$
  - associativity : $c(dA) = (cd)A$



– distributivity : $c(A+B) = cA + cB$, $(c+d)A = cA + dA$.

– linear combinations : $cA + dB$, e.g. if $c=1$ & $d=-1$ we get "$A - B$".

<u>Remark</u>: To be a little more precise about arrows vs. points in real n-space, one would denote <u>points</u> by P, Q, etc. and the arrow from P to Q by $\overrightarrow{PQ}$. If O denotes the origin,



usually one would then write $\overrightarrow{OP}$ as "P", $\overrightarrow{OQ}$ as "Q". The picture then demonstrates that $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ} \Rightarrow P + \overrightarrow{PQ} = Q \Rightarrow$ $\boxed{\overrightarrow{PQ} = Q - P.}$ That is, if $P = (2,2)$ & $Q = (-1,-3)$ then $\overrightarrow{PQ} = (-1,-3) - (2,2) = (-3,-5)$.

Notice that if $P' = (3,1)$ & $Q' = (0,-4)$, then $\overrightarrow{P'Q'} = (0,-4) - (3,1) = (-3,-5)$. So $\overrightarrow{PQ}$ and $\overrightarrow{P'Q'}$ <u>are the same</u> as elements of V_2: all that matters is the n-tuple of real numbers — or intuitively, that the "length and direction" be the same.

<u>Definition</u>: Two vectors A, B are <u>parallel</u> [resp. have the <u>same direction</u>, <u>opposite direction</u>] if $A = cB$ for some $c \in \mathbb{R} \setminus \{0\}$ [resp. $c \in \mathbb{R}^+$, $\mathbb{R}^-$].

• dot product : $A \cdot B := \sum_{k=1}^{n} a_k b_k \in \mathbb{R}$

– commutative : $A \cdot B = B \cdot A$

– distributive : $A \cdot (B + C) = A \cdot B + A \cdot C$

– homogeneous : $c(A \cdot B) = (cA) \cdot B = A \cdot (cB)$

– positive definite : $A \cdot A > 0$ unless $A = 0$ ($\Rightarrow A \cdot A = 0$).

<u>Definition</u>: The <u>length</u> (or <u>norm</u>) of A is $\|A\| := \sqrt{A \cdot A}$.

Notice that $\|cA\|^2 = cA \cdot cA = c^2(A \cdot A) = c^2\|A\|^2 \Rightarrow$
$\|cA\| = |c|\,\|A\|$.

This also gives a notion of distance $d(A,B) = \|\vec{AB}\| = \|B - A\|$.

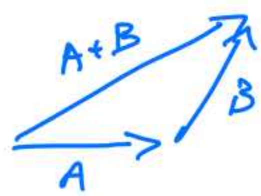
- Cauchy-Schwarz inequality: $|A \cdot B| \leq \|A\|\,\|B\|$
  or (equivalently) $(A \cdot B)^2 \leq (A \cdot A)(B \cdot B)$.

Proof: If $B = 0$ it's obvious. Assume $B \neq 0$ & write

$$0 \leq \left\| \|B\| A - \frac{A \cdot B}{\|B\|} B \right\|^2 = \left(\|B\| A - \frac{A \cdot B}{\|B\|} B \right) \cdot \left(\|B\| A - \frac{A \cdot B}{\|B\|} B \right)$$
$$= \|B\|^2 A \cdot A - \frac{A \cdot B}{\|B\|} \|B\| B \cdot A - \frac{A \cdot B}{\|B\|} \|B\| A \cdot B + \frac{(A \cdot B)^2}{\|B\|^2} B \cdot B$$
$$= \|A\|^2\|B\|^2 - 2(A \cdot B)^2 + (A \cdot B)^2$$
$$= \|A\|^2\|B\|^2 - (A \cdot B)^2.$$

So $(A \cdot B)^2 \leq \|A\|^2\|B\|^2$. Take $\sqrt{\ }$. $\square$

- Triangle inequality: $\|A + B\| \leq \|A\| + \|B\|$.



Proof:
$$\|A+B\|^2 = (A+B) \cdot (A+B)$$
$$= A \cdot A + B \cdot B + 2A \cdot B$$
$$= \|A\|^2 + \|B\|^2 + 2\,A \cdot B \underset{\text{Cauchy-Schwarz}}{\leq} \|A\|^2 + \|B\|^2 + 2\|A\|\|B\|$$
$$= (\|A\| + \|B\|)^2.$$

Take $\sqrt{\ }$'s to conclude. $\square$

- Orthogonality: If $C = A + B$,

The last proof gives $\|C\|^2 = \|A\|^2 + \|B\|^2 + 2A \cdot B$. If A & B are perpendicular we should have $\|C\|^2 = \|A\|^2 + \|B\|^2$ by the Pythagorean Theorem. This motivates the

Definition: $A, B \in V_n$ are called perpendicular (written $A \perp B$) $\iff$ $A \cdot B = 0$.

We'll pick up from here next time.