Lecture 4 : Mathematical Induction

Recall that if we want to prove some assertion for all positive integers, and $A(n)$ denotes the truth of this assertion for the particular positive integer n, then induction says:

$$\left.\begin{array}{l} \text{(i)} \quad A(1) \\ \text{(ii)} \quad A(n) \Rightarrow A(n+1) \\ \qquad (\text{for each } n \in \mathbb{P}) \end{array}\right\} \Longrightarrow A(n) \text{ for every } n \in \mathbb{P}.$$

We begin with a really important consequence of induction:

<u>The Well-Ordering Principle</u>: Every nonempty set $T \subseteq \mathbb{P}$ contains a smallest element. (i.e. $\inf T \in T$)

<u>Proof</u>: Suppose T <u>doesn't have a smallest element</u> ($\inf T \notin T$). Write $S := \{k \in \mathbb{P} \mid k < t \text{ for every } t \in T\}$, and let $\underline{A(n)}$ mean $\underline{n \in S}$.

Clearly $1 \notin T$ (otherwise it's smallest), so $A(1)$ is true.

Suppose $A(n)$ holds. If $A(n+1)$ fails, then $t \leq n+1$ for some $t \in T$. Since T has no smallest element, $\exists\, t' \in T$ with $t' < t$, hence $t' < n+1$. But $n, t' \in \mathbb{P}$, and so $t' \leq n$, contradicting $A(n)$.

So $A(n) \Rightarrow A(n+1)$, making S an inductive set hence $S = \mathbb{P}$, a contradiction since T is nonempty. $\square$

This allows us, in turn, to prove "improved" forms of induction:

$$\left.\begin{array}{l} \text{(i)} \quad A(1) \\ \text{(ii)} \quad A(k) \text{ for all } k \leq n \Rightarrow A(n+1) \\ \qquad [\text{for any } n \in \mathbb{P}] \end{array}\right\} \Longrightarrow A(n) \text{ for every } n \in \mathbb{P}.$$

<u>Proof</u>: Suppose $T := \{n \in \mathbb{P} \mid A(n) \text{ false}\}$ is nonempty. Then by the WOP, T has a least element t_0. For any $n < t_0$, $n \notin T$

— that is, $A(n)$ is true. Now consider the following 2 cases:

$\underline{t_0 = 1}$: impossible, as it contradicts (i)

$\underline{t_0 > 1}$: by (ii), $A(t_0)$ is then true, contradicting $t_0 \in T$.

Therefore our supposition was absurd, and T is empty. $\square$

Here's an application:

Ex/ Define the $\underline{\text{Fibonacci series}}$ by $a_1 = 1$, $a_2 = 2$, &

$a_{n+1} = a_n + a_{n-1}$ (for $n \geq 2$). Let "$A(n)$" mean $a_n < \varphi^n$,

where $\varphi := \frac{1+\sqrt{5}}{2}$ is the golden ratio. Clearly

$A(1)$ holds, since $\varphi > 1$. Suppose that $A(n)$, $A(n-1)$, ...

all hold. Then $a_{n+1} = a_n + a_{n-1} < \varphi^n + \varphi^{n-1} = \varphi^{n+1}$

$\Rightarrow A(n+1)$ holds. (The $\underline{\text{equality}}$ is tricky: it is because

$\varphi^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1 + \varphi$ [now multiply by φ^{n-1}].) //

[$\underline{\text{Remark}}$: In induction, you can always replace (i) by $A(n_0)$ and (ii) by $A(n) \Rightarrow A(n+1)$ for $n \geq n_0$. The conclusion is then that $A(n)$ holds for all $n \geq n_0$. Here n_0 could be 0 or bigger than 1.]

Another application of the WOP (used in Lecture 3!) is:

$\underline{\text{Theorem}}$: If $p, r \in \mathbb{R}$ and $p < r$, then $\exists\, q \in \mathbb{Q}$ with

$$p < q < r.$$

$\underline{\text{Proof}}$: Since $r - p \in \mathbb{R}^+$, the Archimedean property of $\mathbb{R}$ guarantees an $n \in \mathbb{P}$ such that $n(r-p) > 1$ ($\Leftrightarrow nr - np > 1 \Rightarrow np + 1 < nr$).

By the WOP, $S := \{k \in \mathbb{P} \mid k > np\}$ has a least element s_0.
Hence $s_0 > np$, but $s_0 - 1 \le np$, which gives

$$np < s_0 \le np + 1 < nr$$

$$\Rightarrow \quad p < \frac{s_0}{n} < r \quad , \quad \text{done.} \qquad \square$$

One more (a bit long, you can skip if not interested):

<u>Theorem (Euclid)</u>: There are infinitely many prime numbers.

<u>Proof</u>: Suppose $S = \{n \in \mathbb{P} \mid n \ (n \ge 2) \text{ has no prime factorization}\} \neq \emptyset$.
It has a <u>least</u> element N, by WOP. This N can't be prime, b/c then it is its own prime factorization! So N has a divisor $M > 1$, i.e. $N = ML$ for $M, L \in \mathbb{P}$.
But then $M, L < N \Longrightarrow M, L \notin S \Longrightarrow M, L$ have prime factorizations $M_1 \cdots M_k$, $L_1, \ldots, L_\ell \Rightarrow$
$N = M_1 \cdots M_k L_1 \cdots L_\ell$ is a prime factorization $\Rightarrow N \notin S$ ⨳.
So $S = \emptyset$, i.e. every positive integer has a prime factorization.

Now suppose there are finitely many primes $\{P_1, P_2, \ldots, P_m\}$.
Let $Q := 1 + P_1 P_2 \cdots P_m$. It has a prime factorization
$Q = P_1^{a_1} P_2^{a_2} \cdots P_m^{a_m}$, with at least one of the $a_i > 0$.
That is, P_i divides Q; since it also divides $P_1 P_2 \cdots P_m$, it divides $Q - P_1 P_2 \cdots P_m = 1$, a contradiction. $\square$

Final application of induction: recall $\binom{n}{k} := \frac{n!}{(n-k)!\,k!} = \frac{n\cdot(n-1)\cdots(n-k+1)}{k\cdot(k-1)\cdots 1}$

with conventions $\binom{0}{0} = 1$, $\binom{n}{-1} = 0 = \binom{n}{n+1}$. We'll need the

<u>Pascal's triangle identity</u>: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$.

(This doesn't require induction to check. Just write

$$\binom{n}{k}+\binom{n}{k-1} = \frac{n!}{(n-k)!\,k!} + \frac{n!}{(n+1-k)!\,(k-1)!} = \left(\frac{1}{k} + \frac{1}{n+1-k}\right)\frac{n!}{(n-k)!\,(k-1)!}$$

$$= \frac{n+1}{k(n+1-k)}\,\frac{n!}{(n-k)!\,(k-1)!} = \frac{(n+1)!}{(n+1-k)!\,k!} = \binom{n+1}{k}.\quad)$$

(You'll use this in Problem #12 on p. 45 to show what amounts to the statement that $e < 3$.)

<u>Binomial Theorem</u>: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, $a, b \in \mathbb{R}$.

<u>Proof</u>: Well, "$A(1)$" is clearly true:

$$(a+b)^1 = a+b = \binom{1}{0}a + \binom{1}{1}b.$$

"$A(n) \Rightarrow A(n+1)$":

$$(a+b)^{n+1} = (a+b)\cdot(a+b)^n = (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k}$$

reindex the first sum

$$= \sum_{k=1}^{n+1}\binom{n}{k-1}a^k b^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k}$$

$$= \sum_{k=0}^{n+1}\left[\binom{n}{k-1} + \binom{n}{k}\right]a^k b^{n+1-k} = \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n+1-k}.$$

$\square$

<u>P.S.</u>: Have a look at the triangle & Cauchy-Schwarz inequalities on pp. 42-43. (We won't use them right now, though.)