Lecture 40: Linear independence

Let's begin by doing some calculations.

Ex/ $A = (2,-5,-1)$, $B = (-7,-4,6)$

$A \cdot A = 30 \Rightarrow \|A\| = \sqrt{30}$, $B \cdot B = 101 \Rightarrow \|B\| = \sqrt{101}$

$A \cdot B = 0 \Rightarrow A \perp B$

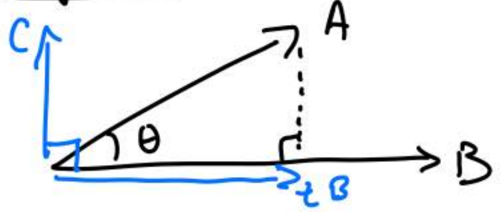
We can produce unit vectors (i.e. length 1) in the directions of A and B by $A/\|A\| = \frac{1}{\sqrt{30}}(2,-5,-1)$, $B/\|B\| = \frac{1}{\sqrt{101}}(-7,-4,6)$.

(Works since $A/\|A\| \cdot A/\|A\| = \frac{A \cdot A}{\|A\|^2} = \frac{\|A\|^2}{\|A\|^2} = 1$.)

The distance between A & B (as points) is

$\|A - B\| = \|(9,-1,-7)\| = \sqrt{131}$. //

<u>Projections</u>



Going down a dimension to V_2, we want to calculate the projection tB of A onto B, as shown.

To do this, write $A = C + tB$, where $C \cdot B = 0$.

This gives $A \cdot B = C \cdot B + tB \cdot B = tB \cdot B \Rightarrow t = \frac{A \cdot B}{B \cdot B} = \frac{A \cdot B}{\|B\|^2}$

$\Rightarrow tB = \left(A \cdot \frac{B}{\|B\|}\right)\frac{B}{\|B\|}$. Moreover, we get $\cos\theta = \frac{t\|B\|}{\|A\|} =$

$\frac{A \cdot B \, \|B\|}{\|B\|^2 \|A\|} = \frac{A}{\|A\|} \cdot \frac{B}{\|B\|}$.

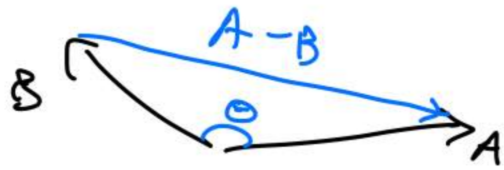
by Cauchy-Schwarz, this is always between -1 & 1 — so is cosine of something!

More generally, these formulas are taken as the <u>definitions</u> of projection & angle in higher dimensions.

Ex/ $A = (1,2,1)$, $B = (1,1,0)$ $\leadsto$ $\cos\theta = \dfrac{(1,2,1)\cdot(1,1,0)}{\|(1,2,1)\|\,\|(1,1,0)\|} = \dfrac{3}{\sqrt{6}\sqrt{2}} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow$ $\theta = \pi/6$ (always taken to be in $[0,\pi]$). //

The law of cosines is an immediate consequence of our definition of angle: $\boxed{\|A-B\|^2 = \|A\|^2 + \|B\|^2 - 2\|A\|\|B\|\cos\theta}$



(Try it!)

Linear Combinations

If $A_1, \dots, A_k \in V_n$, a linear comb. of them is $\sum_i c_i A_i$ ($c_i \in \mathbb{R}$).

Their span (or linear span) is the set of all of these linear combinations. If this span equals V_n, we say "the $\{A_i\}$ span V_n".

Ex/ $\{E_1, E_2, \dots, E_n\}$ span V_n, where $E_i := (0, \dots, 0, \underset{\substack{\uparrow \\ i^{\text{th}} \text{ coordinate}}}{1}, 0, \dots, 0)$ are the "unit coordinate vectors". Any

A can be written as a linear combination of them (in a unique way, as it turns out):

$$A = (a_1, a_2, \dots, a_n) = a_1(1,0,\dots,0) + a_2(0,1,0,\dots,0) + \dots + a_n(0,\dots,0,1)$$
$$= \sum_{i=1}^{n} a_i E_i \ .$$

//

Notation: if $S = \{A_1, \dots, A_k\}$, write $L(S)$ for the linear span.

S spans $V_n \iff L(S) = V_n$.

Definition: A set S of vectors is linearly independent if

$$\sum_{i=1}^{k} c_i A_i = 0 \implies \text{all } c_i = 0 .$$

(The book calls this "spanning the 0-vector uniquely". An immediate consequence is that S spans every vector in $L(S)$ uniquely if the sense that $\sum c_i A_i = \sum d_i A_i \implies c_i = d_i \ (\forall i).$)

Ex/ $S = \{(1,2),(1,0),(0,1)\}$ spans V_2 but is $\overbrace{\text{not independent}}^{\text{dependent}}$

$S = \{(1,0)\}$ doesn't span V_2 but is independent

$S = \{(1,0),(1,2)\}$ spans V_2 & is independent

Note: if a set contains the 0-vector, it is not independent. (Why?) //

Definition: S is a basis of V_n if it spans V_n and is linearly independent.

Theorem: (i) Any basis of V_n consists of exactly n vectors.

(ii) Any set of linearly independent vectors is a subset of a basis.

Corollary: Any set of n linearly independent vectors is a basis.

To prove the Theorem, we'll use the following

Lemma: If $S = \{A_1, \dots, A_k\} \subset V_n$ is linearly independent, then any set of $k+1$ vectors in $L(S)$ is dependent.

Proof: ($k=1$) $S = \{A_1\}$, $A_1 \neq O$, $L(S)$ consists of multiples of A_1, so any two are dependent ($B_i = c_i A_1 \Rightarrow c_1 B_2 - c_2 B_1 = 0$)

(inductive step) $T = \{B_1, \dots, B_{k+1}\} \subset L(S)$, $B_i = \sum_{j=1}^{k} a_{ij} A_j$.

assume for $k-1$, prove for k

If $a_{i1} = 0 \ (\forall i)$, then $T \subset L(\{A_2, \dots, A_k\}) \xrightarrow[\text{(k-1)}]{\text{inductive assumption}} T$ dependent

Otherwise, some $a_{i1} \neq 0$ — we may assume $a_{11} \neq 0$.

Writing $c_i = \frac{a_{i1}}{a_{11}}$,

$$c_i B_1 = a_{i1} A_1 + \sum_{j=2}^{k} c_i a_{1j} A_j$$
$$-\left(B_i = a_{i1} A_1 + \sum_{j=2}^{k} a_{ij} A_j\right)$$
$$c_i B_1 - B_i = \sum_{j=2}^{k} (c_i a_{1j} - a_{ij}) A_j$$

$\Rightarrow \{c_i B_1 - B_i\} \subset L(\{A_2, \dots, A_k\}) \xrightarrow[\text{assumption}]{\text{inductive}} \{c_i B_1 - B_i\}$ dependent

$$\Rightarrow \sum_{i=2}^{k+1} t_i (c_i B_1 - B_i) = 0 \text{ for some } t_i \in \mathbb{R} \Rightarrow \left(\sum_{i=2}^{k+1} t_i c_i\right) B_1 - \sum_{i=2}^{k+1} t_i B_i = 0$$

$\Rightarrow T$ is dependent. □

Proof of Theorem: (i) $V_n = L(\underbrace{\{E_1, \ldots, E_n\}}_{\text{this is one basis of } V_n})$. By the lemma, if S consists of more than n vectors, it is dependent (hence not a basis). If S consists of less than n vectors, it can't span V_n — otherwise, the lemma would say $E_1, \ldots, E_n$ are dependent. So if S is to be a basis, it had better consist of exactly n vectors

(ii) $S = \{A_1, \ldots, A_k\}$ independent. If it doesn't span V_n, pick A_{k+1} not in $L(S)$. $\{A_1, \ldots, A_{k+1}\}$ is still independent (why?). Continue in this fashion until you have n elements. At this point the set spans V_n — otherwise we'd get $n+1$ independent vectors at the next step, which is absurd by the lemma (since $\{E_1, \ldots, E_n\}$ spans V_n). □

Ex/ Let $S = \{A_1, \ldots, A_k\}$ be orthogonal, i.e. $A_i \cdot A_j = 0$ $\forall i \neq j$.

nonzero vectors

Then S is independent: if $0 = \sum c_i A_i$ then taking $A_j \cdot$ both sides gives $0 = c_j \|A_j\|^2 \Rightarrow c_j = 0$ $(\forall j)$. (In particular, if $k = n$ then S is a basis of V_n.) For any $B \in L(S)$, we have

$$B = \sum_{i=1}^{k} b_i A_i \xrightarrow{A_j \cdot} A_j \cdot B = b_j A_j \cdot A_j \Rightarrow b_j = \frac{A_j \cdot B}{A_j \cdot A_j}$$

$\Rightarrow B = \sum_{i=1}^{k} \frac{B \cdot A_i}{A_i \cdot A_i} A_i$ is the unique linear combination of the $\{A_i\}$ giving B. //