Lecture 41: Lines & planes in n-space

LINES

"lines" are all sets of this form

Given $\underset{\text{point}}{P \in V_n}$, $\underset{\text{direction vector}}{A \in V_n \setminus \{0\}}$, set $L(P,A) := \{P + tA \mid t \in \mathbb{R}\} \subset V_n$.

This is the linear span $L(A)$ translated by P: $L(P,A) = \underbrace{L(A)}_{\text{linear span}} + P$.

It is parametrized by (= the image of)

(parametric form) $X : \mathbb{R} \to V_n$

$t \longmapsto X(t) := \underbrace{(p_1, \dots, p_n)}_{P} + t\underbrace{(a_1, \dots, a_n)}_{A} = (p_1 + ta_1, \dots, p_n + ta_n)$

Facts:

- A line ℓ contains $P \iff \ell = L(P,A)$ for some A.

  Proof: ($\Leftarrow$) obvious

  ($\Rightarrow$) Given $\ell = L(Q,B) \ni P$, write $\ell' = L(P,B)$. I claim $\ell = \ell'$.
  Indeed, $P \in \ell \Rightarrow P = Q + t_0 B$. So $X \in \ell \iff X = Q + t_X B \iff$
  $X = (P - t_0 B) + t_X B = P + (t_X - t_0)B = P + t_X' B \iff X \in \ell'$. $\square$

- $L(P,A) = L(P,B) \iff A \parallel B$. (and $A \parallel B \Rightarrow L(Q,A) = L(Q,B)$).

  Definition: $L(P,A) \parallel L(Q,B) \overset{\text{def.}}{\iff} A \parallel B$.

- If $P \neq Q$, then $L(P, Q-P)$ is the unique line containing P & Q. (Reduce to the case $P = 0$; then a line containing Q contains $L(Q)$ hence $= L(Q)$.)

- $\{A, B\} \subset V_n$ is linearly independent $\iff$ A, B don't lie on a line thru 0
  ($\iff A, B \neq 0$ and A & B not parallel)

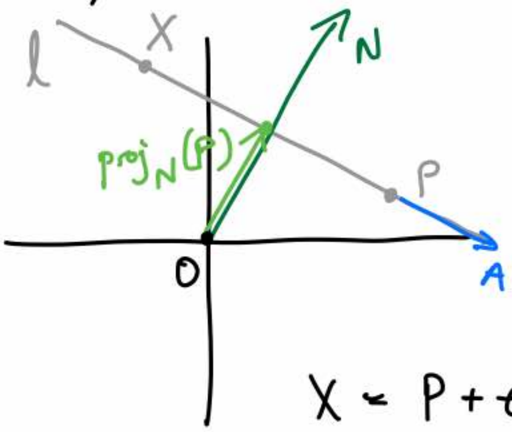
  Proof: ($\Rightarrow$) If they lie on a line ℓ thru 0, then $\ell = L(0,C) = L(C) \Rightarrow$
  $A = t_0 C$, $B = t_1 C \Rightarrow t_1 A - t_0 B = 0 \Rightarrow$ dependent (not L.I.)

  ($\Leftarrow$) If A, B dependent, then $B = cA$ (or vice versa) $\Rightarrow$ both on $L(0,A)$. $\square$

Ex/ Do $P = (2,1,1)$, $Q = (4,1,-1)$, $R = (3,-1,1)$ lie on a straight line?

(Hint: equiv. to same question for 0, $Q-P$, $R-P$.) //

Ex/ $(n=2)$ Let $\ell = (P, A)$, $A = (a_1, a_2)$; and consider



$N := (a_2, -a_1)$. Since $N \cdot A = 0$, $N \perp A$.

Let $\mathcal{L} := \{ X \in V_2 \mid \underbrace{(X-P)\cdot N = 0}_{\text{(Cartesian form)}} \}$.

Then $\ell = \mathcal{L}$ because $X \in \ell \iff$

$$X = P + tA \iff X - P = tA \iff (X-P)\cdot N = 0.$$

Now write $\operatorname{proj}_N(P) := \left(P \cdot \frac{N}{\|N\|}\right)\frac{N}{\|N\|} \in L(N)$. This is on ℓ because $\left(\frac{P\cdot N}{N\cdot N} N - P\right)\cdot N = \cancel{\frac{P\cdot N}{N\cdot N} N\cdot N} - P\cdot N = 0$. We claim that $\|\operatorname{proj}_N(P)\| = \frac{P\cdot N}{\|N\|}$ is the distance from O to ℓ:

- Cauchy-Schwarz proof: $\|X\| = \frac{\|X\|\|N\|}{\|N\|} \overset{\text{C-S}}{\geq} \frac{|X\cdot N|}{\|N\|} \overset{\text{eqn. of } \mathcal{L}}{=} \frac{|P\cdot N|}{\|N\|}$ for any $X \in \ell$ $\square$
- Calculus proof: set $0 = \frac{d}{dt}|X(t)|^2 = \frac{d}{dt} X(t)\cdot X(t) = 2X'(t)\cdot X(t)$ (why?)
  $(X(t) := P + tA)$
  $= 2A\cdot X(t)$. If $t = t_0$ solves this, then
  $X(t_0) \perp A \implies X(t_0) \parallel N \implies X(t_0) = L(N) \cap \ell = \operatorname{proj}_N(P)$. $\square$

More generally, if we want $\operatorname{dist}(Q, \ell)$ for Q different from O, just translate the line to $\ell - Q = L(P-Q, A)$, so that
$\operatorname{dist}(Q,\ell) = \operatorname{dist}(O, L(P-Q, A)) = \|\operatorname{proj}_N(P-Q)\| = \frac{|(P-Q)\cdot N|}{\|N\|}$. That is, you take any point $P \in \ell$ and dot $P - Q$ with the unit normal vector $\frac{N}{\|N\|}$.

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Planes

Given $P \in V_n$, $\{A,B\} \subset V_n$ independent, set

$$M(P,\{A,B\}) := \{P + sA + tB \mid s,t \in \mathbb{R}\} \underset{(*)}{=} \overbrace{L(\{A,B\})}^{\text{linear span}} \overbrace{+\,P}^{\text{translation}}.$$

A plane is any subset of V_n of this form; we may regard it as the image of a parametrization map

$$X : \mathbb{R}^2 \longrightarrow V_n$$
$$(s,t) \longmapsto X(s,t) := P + sA + tB = (p_1 + sa_1 + tb_1, \dots, p_n + sa_n + tb_n).$$

Two planes $M(P,\{A,B\})$, $M(Q,\{C,D\})$ are called parallel if $L(\{A,B\}) = L(\{C,D\})$.

Facts:

- $M(P,\{A,B\}) = M(P,\{C,D\}) \iff L(\{A,B\}) = L(\{C,D\})$ [just use $(*)$]
- $M(P,\{A,B\}) = M(Q,\{A,B\}) \iff Q \in M(P,\{A,B\})$
  [again, $(*)$ gives LHS $\iff L(\{A,B\}) + P = L(\{A,B\}) + Q \iff Q - P \in L(\{A,B\}) \iff$
  $Q \in P + L(\{A,B\}) = M(P,\{A,B\})$]
- If P, Q, R aren't collinear (don't lie on a line), then $O, Q-P, R-P$ are non-collinear $\implies$ $Q-P$ & $R-P$ are not parallel $\implies \{Q-P, R-P\}$ independent.
  So we may define $M_{P,Q,R} := M(P, \{Q-P, R-P\})$, which contains P, Q, R.

  Theorem: $M_{P,Q,R}$ is the only plane containing P, Q, R. ("3 non-collinear pts. determine a plane")

  Proof: $\{\text{planes containing } P,Q,R\} = \{\text{planes containing } O, Q-P, R-P\} + P$.
  So we may assume $P = O$. We need then to show that $M := L(\{Q,R\})$ is the only plane containing O, Q, R, given $\{Q,R\}$ linearly independent.
  Let $M' = M(O,\{A,B\}) = L(\{A,B\})$ also contain O, Q, R.
  Then $Q = aA + bB$ & $R = cA + dB \implies L(\{R,Q\}) \subseteq L(\{A,B\})$.
  $\cdot d$: $dQ = adA + bdB$ & $\cdot b$: $bR = bcA + bdB \overset{\text{subtract}}{\implies} dQ - bR = (ad - bc)A$

$\Rightarrow A = \frac{d}{ad-bc} Q + \frac{-b}{ad-bc} R \in L(\{R,Q\})$. (We know $ad-bc \neq 0$ b/c otherwise $dQ - bR = 0 \Rightarrow \{Q,R\}$ dependent.) Similarly, $B \in L(\{R,Q\})$; & so $L(\{A,B\}) \subseteq L(\{R,Q\})$. So $M' = M$, done. $\square$

- Corollary: 3 vectors A, B, C in V_n are dependent $\Leftrightarrow$ they lie on the same plane thru the origin O.

Proof: ($\Rightarrow$) wolog $C = aA + bB \Rightarrow C \in L(\{A,B\})$ †

($\Leftarrow$) wolog wma A, B independent; then $\exists!$ plane thru O (namely $L(\{A,B\})$) containing A, B (by Thm.). Hence $C \in L(\{A,B\})$. $\square$

Ex/ Consider the plane $M = M((1,1,-1), \{(2,2,3),(2,-2,-1)\}) \subset V_3$. Which of the points $(2,2,\frac{1}{2})$, $(4,0,-\frac{1}{2})$, $(-3,1,-3)$, $(3,1,3)$, $(0,0,0)$ lie on M? (Hint: write M parametrically, then eliminate s & t to get a Cartesian equation.) //

Next time, we will do the plane analogue of the normal vector business (done above for lines). The key point is to introduce the cross-product to define N by $N := (Q-P) \times (R-P)$. This allows us to (for example) find the distance from a point to a plane.

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† WARNING: $\{A,B,C\}$ dependent does not imply that each one is a linear combination of the others. It only implies that at least one of them is a linear combination of the other two. (Why?)