Lecture 42: The cross-product

We will work entirely in V_3 for this lecture. Let $\begin{cases} A = (a_1, a_2, a_3) \\ B = (b_1, b_2, b_3) \\ C = (c_1, c_2, c_3) \end{cases}$

<u>Definition:</u> (i) $B \times C := (b_2c_3 - b_3c_2,\ b_3c_1 - b_1c_3,\ b_1c_2 - b_2c_1) = \underbrace{\begin{vmatrix} E_1 & E_2 & E_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}_{\text{determinant notation (will review in class)}}$

where $E_1 = (1,0,0)$, $E_2 = (0,1,0)$, $E_3 = (0,0,1)$ are sometimes called i, j, k. It's clear at once that $C \times B = -B \times C$.

(ii) $A \cdot (B \times C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is the scalar triple-product.

more compact notation: $[A\,B\,C]$

From this it's immediate that $B \cdot (B \times C) = 0 = C \cdot (B \times C)$, so $B \times C$ is $\perp$ to B & C.

<u>Theorem 1:</u> (i) $\|A \times B\| = \|A\|\,\|B\|\,|\sin\theta|$ (where θ is the angle between A & B)

= area of parallelogram spanned by A & B

(ii) $|A \cdot (B \times C)|$ = volume of parallelepiped spanned by A, B, C

<u>Proof:</u> (i) $\|A \times B\|^2 + (A \cdot B)^2 = (A \times B)\cdot(A \times B) + \left(\sum a_i b_i\right)^2$

$$= \sum_{i<j} (a_i b_j - a_j b_i)^2 + \sum_i a_i^2 b_i^2 + \sum_{i \neq j} a_i b_i a_j b_j$$

$$= \sum_{i<j} a_i^2 b_j^2 + \sum_{i<j} a_j^2 b_i^2 - \cancel{2\sum_{i<j} a_i b_i a_j b_j} + \sum_i a_i^2 b_i^2 + \cancel{2\sum_{i<j} a_i b_i a_j b_j}$$

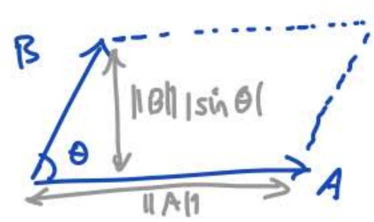
$$= \sum_{i \neq j} a_i^2 b_j^2 + \sum_i a_i^2 b_i^2$$

$$= \left(\sum_i a_i^2\right)\left(\sum_j b_j^2\right) = \|A\|^2 \|B\|^2$$

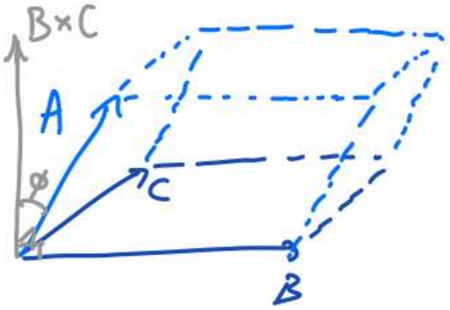
since $A \cdot B = \|A\|\,\|B\| \cos\theta$

$\|A \times B\|^2 = \|A\|^2\|B\|^2 - \|A\|^2\|B\|^2\cos^2\theta$

$= \|A\|^2\|B\|^2 \sin^2\theta$. Now take square root.



(ii)



The volume of the solid, since the cross-sectional area is constant ($=$ area(base)) is area(base) $\cdot$ height $= \|B\times C\|\cdot\|A\|\,|\cos\phi|$
$= |A\cdot(B\times C)|$. $\square$

Since the parallelepiped doesn't depend on the order of the vectors, we see that $A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B)\ \big(=(A\times B)\cdot C\big)$.

(OK, there is a question of what the sign is: either appeal to the "right-hand rule" or wait until we learn more about determinants.)
↳ do example

Before proceeding to the next result, recall that for the proof of the Cauchy-Schwarz inequality $|A\cdot B| \le \|A\|\,\|B\|$ we used

$$0 \le \left\| \|B\| A - \frac{A\cdot B}{\|B\|} B \right\|^2 = \dots = \|A\|^2\|B\|^2 - (A\cdot B)^2.$$

Therefore (for $A, B \neq 0$) we have equality in C-S $\iff \|B\| A = \frac{A\cdot B}{\|B\|} B \iff A \parallel B$.

Theorem 2: (i) $\{A,B\}$ independent $\iff A\times B \neq 0 \implies \{A, B, A\times B\}$ independent.

(ii) Assume $\{A,B\}$ L.I. Then $N \perp A, B \implies N \parallel A\times B$

(iii) $\{A,B,C\}$ independent $\iff A\cdot(B\times C) \neq 0$

Proof: (i) By Theorem 1(i), $\|A\times B\| = \|A\|\,\|B\|\,|\sin\theta|$ is zero $\iff$ A or B is 0 or $A \parallel B$ ($\Leftrightarrow \sin\theta = 0$) $\iff \{A,B\}$ dependent.

Now suppose $\{A,B\}$ independent, so that $\|A\times B\| \neq 0$, and write $aA + bB + c(A\times B) = 0$. Dotting with $A\times B$ gives

$$0 + 0 + c\,\|A\times B\|^2 = 0 \quad\longrightarrow\quad c = 0$$

$\longrightarrow aA + bB = 0 \underset{A,B \text{ indep}}{\longrightarrow} a = b = 0$. So $\{A, B, A\times B\}$ are independent.

(ii) Since $\{A, B, A\times B\}$ L.I. by (i), they are a basis of V_3.

so $N = aA + bB + c(A\times B) \overset{\cdot(A\times B)}{\Longrightarrow} N\cdot(A\times B) = 0 + 0 + c\|A\times B\|^2$

$\overset{\cdot N}{\Longrightarrow} \|N\|^2 = 0 + 0 + c\, N\cdot(A\times B)$

$\Longrightarrow \underbrace{\|N\|^2}\|A\times B\|^2 = \underbrace{(N\cdot(A\times B))\, c}\,\overbrace{c\,\|A\times B\|^2} = (N\cdot(A\times B))\overbrace{(N\cdot(A\times B))}$

$\overset{\sqrt{\cdot}}{\Longrightarrow} \|N\|\,\|A\times B\| = |N\cdot(A\times B)|$ is equality in Cauchy-Schwarz

$\Longrightarrow N \parallel A\times B$.

(iii) ($\Longleftarrow$): Suppose $\{A,B,C\}$ dependent. Then either

- $\{B,C\}$ dependent $\underset{(i)}{\Longrightarrow} B\times C = 0$
- $\{B,C\}$ independent & $A = bB + cC \Longrightarrow A\cdot(B\times C) = 0$ since $B, C \perp B\times C$

($\Longrightarrow$): Suppose $A\cdot(B\times C) = 0$. If $B\times C = 0$, then $\{B,C\}$ dependent by (i) $\Rightarrow \{A,B,C\}$ dependent, done.

If $B\times C \neq 0$, then $\{B,C\}$ indep. $\Rightarrow \{B, C, B\times C\}$ basis

$\Longrightarrow A = a(B\times C) + bB + cC$

$\overset{\cdot(B\times C)}{\Longrightarrow} 0 = A\cdot(B\times C) = a\|B\times C\|^2 + 0 + 0 \Longrightarrow a = 0$

$\Longrightarrow A = bB + cC \Longrightarrow \{A, B, C\}$ dependent. $\square$

<u>Application to linear systems</u>. Consider $\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}$ with $\{A,B,C\}$ <u>independent</u>.

Then there <u>exists</u> a <u>unique</u> solution in (x,y,z): rewrite the system as $xA + yB + zC = D$. Taking dot products gives

$$\left.\begin{aligned} \cdot(B\times C):&\quad x[ABC] + 0 + 0 = [DBC] \rightsquigarrow x = [DBC]/[ABC] \\ \cdot(C\times A):&\quad 0 + y[BCA] + 0 = [DCA] \rightsquigarrow y = [ADC]/[ABC] \\ \cdot(A\times B):&\quad 0 + 0 + z[CAB] = [DAB] \rightsquigarrow z = [ABD]/[ABC]. \end{aligned}\right\} \text{"Cramer's rule"}$$

<u>Uniqueness</u>: if $xA + yB + zC = D = x'A + y'B + z'C$, then

$$\left.\begin{aligned} &(x-x')A + (y-y')B + (z-z')C = 0 \\ &\& \ \{A,B,C\} \text{ lin. independent} \end{aligned}\right\} \Longrightarrow \begin{aligned} &x-x', y-y', z-z' = 0 \\ &\text{i.e. } (x',y',z') = (x,y,z). \end{aligned}$$