Lecture 43: Conic sections

These arise in many contexts:

- plane sections of the cone $x^2+y^2=z^2$
- definition via focal points

Apostol "ice-cream-cone proof"

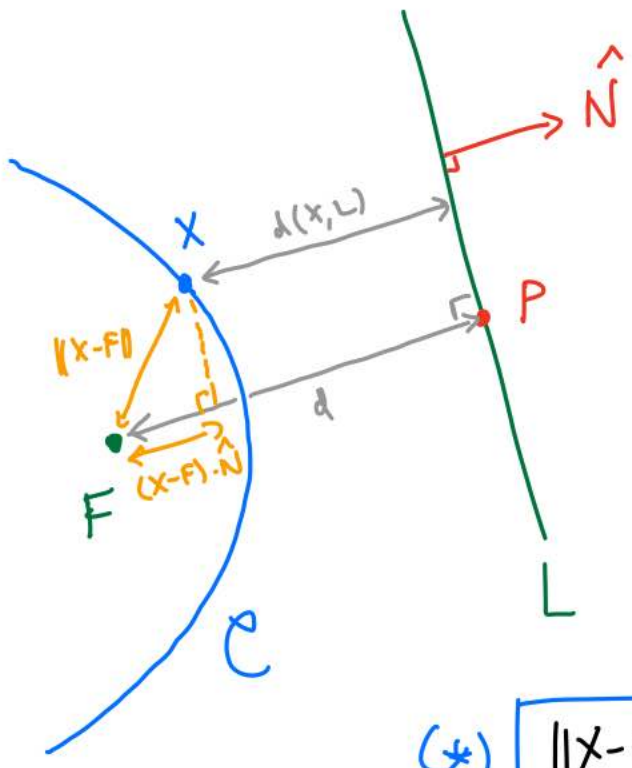
- definition via "directrix" (see below)
- solution sets to quadratic polynomials in (x,y)

today

- trajectories of particle in gravitational field (Ch. 14)

<u>Definition</u>: Given $\underset{\text{eccentricity}}{e} \in \mathbb{R}^+$, $\underset{\text{focal pt.}}{F} \in V_2$, $\underset{\text{directrix}}{L} \subset V_2$ line not containing F, let $\mathcal{C}_{e,F,L} := \{ X \in V_2 \mid \|X-F\| = e\, d(X,L) \}$. This is called an <u>ellipse</u> if $e<1$, a <u>parabola</u> if $e=1$, and a <u>hyperbola</u> if $e>1$.

Note that $d(X,L) = |(X-P)\cdot \hat{N}|$, for any $P \in L$ and $\hat{N}$ = unit vector normal to L.



For convenience we choose $\hat{N}$ so that $\hat{N}\cdot(F-P)<0$, and P to minimize $d(F,P)$. Namely, Cauchy-Schwarz $\Rightarrow$
$\|F-P\| \geq |\hat{N}\cdot(F-P)| = d$ (indep. of p)
has equality iff $F-P \parallel \hat{N}$ $\Rightarrow$ $P = F + d\hat{N}$.
The sign of $(X-P)\cdot\hat{N}$ then decides which side of L (or "branch of $\mathcal{C}$") we are on.

With these choices, the equation becomes

$$(*) \quad \boxed{\|X-F\| = e\,|(X-F)\cdot\hat{N} - d|.} \qquad \text{(why?)}$$

Polar form.

Choose coordinate system so that $F = 0$, $L = \overbrace{\{x = d\}}^{\text{vertical}}$, $N = (1,0)$, and write $X = (x,y) = (r\cos\theta, r\sin\theta)$. (*) becomes

$$\|X\| = e\left|X\cdot\hat{N} - d\right|$$

i.e. $$r = e\left|r\cos\theta - d\right|.$$

left branch: $(X - (d,0))\cdot\hat{N} < 0 \iff r\cos\theta < d$

$$r = ed - er\cos\theta \implies r = \frac{ed}{e\cos\theta + 1}$$

right branch: $r\cos\theta > d$

$$r = er\cos\theta - ed \implies r = \frac{ed}{e\cos\theta - 1}$$

Since $r > 0$, the equation forces $e > 1$: so there is only a "right branch" for hyperbolas.

Cartesian form for parabolas. $(e = 1)$

$P = (0,-c)$

Choosing coordinates so that $F = (0, c)$ and $\hat{L} = \{y = -c\}$,

$$\|X - F\| = d(X, L)$$

$(\cdot)^2$

$$(X-F)\cdot(X-F) = (y+c)^2$$

$$(x, y-c)\cdot(x, y-c) = (y+c)^2$$

$$x^2 + (y-c)^2 = (y+c)^2$$

$$x^2 = 4cy$$

$$y = x^2/4c.$$

Yep, that's a parabola.

Cartesian form for ellipses & hyperbolas. ($e \neq 1$)

We don't want $F = 0$ here: would rather have symmetry about the origin. Equation (*) expands as

$$\|X - F\| = e\,|X \cdot \hat{N} - F \cdot \hat{N} - d| = |eX \cdot \hat{N} - a|, \quad \underset{(\dagger)}{a = ed + eF \cdot \hat{N}}$$

$(\cdot)^2$

$$\|X\|^2 - 2X \cdot F + \|F\|^2 = e^2 (X \cdot \hat{N})^2 - 2ea\, X \cdot \hat{N} + a^2$$

want $-X$ to also satisfy this whenever X does:

$$\|X\|^2 + 2X \cdot F + \|F\|^2 = e^2 (X \cdot \hat{N})^2 + 2ea\, X \cdot \hat{N} + a^2$$

$$\Rightarrow \quad X \cdot F = ea\, X \cdot \hat{N} \quad (\forall X) \quad \Rightarrow \quad F = ea\hat{N}$$

$$\Rightarrow \quad F \cdot \hat{N} = ea \overset{(\dagger)}{\Rightarrow} a = ed + e^2 a \Rightarrow e \neq 1 \text{ and } \begin{cases} a = \dfrac{ed}{1-e^2} \\ F = \dfrac{e^2 d}{1-e^2}\hat{N} \\ \quad (= ea\hat{N}) \end{cases}$$

$$\Rightarrow \text{eqn. is } \boxed{X \cdot X + e^2 a^2 = e^2 (X \cdot \hat{N})^2 + a^2.} \quad (**)$$

In this scenario, define $-F$ as the second focal point.

Remark: It may appear that we have imposed conditions that give <u>special</u> conics. In fact, this isn't really the case. Given an arbitrary $e \neq 1$ conic, replacing X by $X' = X - T$ effects a <u>translation</u> of $\mathcal{C}$, L, & F and doesn't affect $\hat{N}$. In (*) this yields

$$\|X' + T - F\| = e\,|(X' + T - F) \cdot \hat{N} - d|$$

$$\|X' - F'\| = e\,|(X' - F') \cdot \hat{N} - d| \quad \text{where } F' = F - T.$$

We want to have $F' = eaN$; to get this, simply choose $T = F - eaN$.

To simplify (**) further, "rotate" coordinates so that $N = (1, 0)$.

$$(\Rightarrow \pm F = (\pm ea, 0)) \Rightarrow d = \frac{(1-e^2)a}{e} = \frac{a}{e} - ea$$

$$d + ea = a/e \qquad \Rightarrow x = \frac{a}{e} \text{ is the directrix.)}$$

(**) becomes $x^2+y^2+e^2a^2 = e^2x^2+a^2$

$$x^2(1-e^2)+y^2 = a^2(1-e^2)$$

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)} = 1}$$

$e<1$: ellipse with axis lengths 

$e>1$: hyperbola with asymptotes $y = \pm\sqrt{e^2-1}\,x$

(Remark : taking $e\to 0$ we obtain a circle.)

Hyperboles can be characterised as the set of points whose difference of its distances to the focal points remains constant. Let's do this for the "left" branch of a hyperbola symmetric about O with foci $\pm F$. We need to show that $\|X-F\|-\|X+F\|$ is constant. In fact this is

$$= |eX\cdot\hat{N}-a| - |eX\cdot\hat{N}+a| = |e|\left\{\Big|\underbrace{\overbrace{(X-F-d\hat{N})}^{X-P}\cdot\hat{N}}_{\substack{\text{always negative}\\ \text{(left branch)}}}\Big| \Longrightarrow - \Big|\underbrace{(X+F+d\hat{N})\cdot\hat{N}}_{\substack{\text{always negative}\\ \text{since } (F+d\hat{N})\cdot\hat{N} = \\ \frac{e^2d}{1-e^2}+d = \frac{d}{1-e^2} < 0}}\Big|\right\}$$

$$= -(eX\cdot\hat{N}-a) + (eX\cdot\hat{N}+a)$$

$$= 2a, \text{ which is constant.}$$