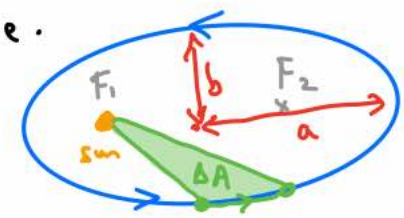
Lecture 44: Vector Calculus

In the first quarter of the 17^{th} Century, through his study of astronomical data collected by Tycho Brahe, Kepler was led to propose the following three Laws of Planetary Motion:

1. The motion is elliptical (not circular, as Copernicus had it), with the sun as one of the foci of the ellipse.



2. Area is swept out at a constant rate (i.e. dA/dt = const. ; see the figure)
3. The period of revolution (time to go once around the sun) is proportional to $a^{3/2}$, where a = length of semimajor axis.

One of the first great triumphs of "The Calculus" was Newton's proof, nearly 70 years later, that Kepler's laws follow from his 2^{nd} law ($\vec{F}(t) = m\,\vec{a}(t)$, where $\vec{F}$ is the vector sum of all the forces acting on an object & $\vec{a}$ is its acceleration) and Law of Universal Gravitation.

To appreciate (among other things) this profound result, we turn to the calculus of vector-valued functions

interval $\subseteq \mathbb{R}$ (possibly all of $\mathbb{R}$)

$$F = (f_1, \dots, f_n) : I \longrightarrow V_n$$

components of F

$$x \longmapsto (f_1(x), \dots, f_n(x)),$$

usually written $F(x)$ (as here) or $X(t)$. We shall also write

$$\mathcal{C} := F(I) = \{ F(x) \mid x \in I \} \subset V_n$$

for the "image curve" parametrized (traced out) by F.

Operations on vector-valued functions [Given F, G vector-valued & h scalar-valued ...]

- $F+G$, hF, $F\times G$ (3-space), $F\cdot G$, $(F\circ u)(x) := F(u(x))$

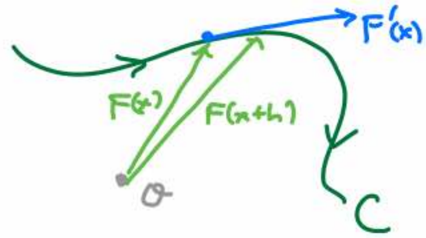
  define new vv fcns. — defines a scalar-valued fcn. — If F parametrizes C, then so does this ("reparametrization")

- $F'(x) := \lim_{h\to 0}\frac{1}{h}\left(F(x+h)-F(x)\right) = \left(\lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}, \dots, \lim_{h\to 0}\frac{f_n(x+h)-f_n(x)}{h}\right) = \left(f_1'(x), \dots, f_n'(x)\right)$

- $\int_a^b F(x)\,dx = \left(\int_a^b f_1(x)\,dx, \dots, \int_a^b f_n(x)\,dx\right)$,

  limits, derivatives, & integrals all <u>componentwise</u>

  $\int F(x)\,dx = \left(\int f_1(x)\,dx, \dots, \int f_n(x)\,dx\right) + (C_1, \dots, C_n)$



Basic properties of the operations

- $\int (c_1F + c_2G)\,dx = c_1\int F\,dx + c_2\int G\,dx$, $\int C\cdot F\,dx = C\cdot\int F\,dx$ ($C = (c_1, \dots, c_n)$; $\cdot$ = dot product)
- $(F\cdot G)' = F'\cdot G + F\cdot G'$, similar for hF, $F\times G$; $(F\circ u)' = u'(F'\circ u)$
- <u>FTC</u>: $\frac{d}{dx}\int_a^x F(t)\,dt = F(x)$, $\int_a^b F'(t)\,dt = F(b) - F(a)$ (Why? b/c true componentwise)

  integral of velocity = change in position

⊙ $\left\| \int_a^b F(t)\,dt \right\| \le \int_a^b \|F(t)\|\,dt$

<u>Proof</u>: $\|X\|^2 = X\cdot X = X\cdot\int_a^b F(t)\,dt = \int_a^b X\cdot F(t)\,dt \le \int_a^b |X\cdot F(t)|\,dt$

$\underset{\text{Cauchy–Schwarz}}{\le} \int_a^b \|X\|\,\|F(t)\|\,dt = \|X\|\int_a^b \|F(t)\|\,dt$. Now cancel. □

⊙ (motion on a circle/sphere/etc.) If $\|F(t)\| = $ cst., then $F \perp F'$ everywhere.

<u>Proof</u>: $0 = \frac{d}{dt}\|F(t)\|^2 = \frac{d}{dt}F\cdot F = F'\cdot F + F\cdot F' = 2F\cdot F'$. □

<u>Curvilinear motion</u>. As you can see, I prefer to work with t as variable; so I now switch to $X(t) = (x_1(t), \dots, x_n(t)) =$ "position", $X'(t) =:$ velocity, $\|X'(t)\| =$ speed, $X''(t) =:$ acceleration (sometimes also written $\vec{r}(t)$, $\vec{v}(t)$, $v(t)$, $\vec{a}(t)$ respectively. One imagines a particle

moving along $\mathcal{C} = X(I)$, whose instantaneous direction of motion at $t = t_0$ is $X'(t_0)$.

<u>Definition</u>: (i) If $X'(t_0) \neq 0$, then the line $L(X(t_0), X'(t_0))$ is called the <u>tangent line</u> to $\mathcal{C}$ at $X(t_0)$.

(ii) If $X'(t_0) \neq 0$, then $T(t_0) := \dfrac{X(t_0)}{\|X'(t_0)\|}$ is called the <u>unit tangent vector</u>.

(iii) $\mathcal{C}$ is <u>smooth</u> at $X(t_0) \iff \lim_{t \to t_0} T(t_0)$ exists. ($\mathcal{C}$ is "smooth" if it is smooth at every point; roughly, this means it has no corners or cusps.)

(you can't define tangent line at such points)

<u>Remarks</u>: (1) Reparametrization $X(t) \rightsquigarrow X(u(s))$ ($t_0 = u(s_0)$) merely changes $L(X(t_0), X'(t_0))$ to $L(X(u(s_0)), u'(s_0) X'(u(s_0))) = L(X(t_0), u'(s_0) X'(t_0)) = L(X(t_0), X'(t_0))$.

parallel!

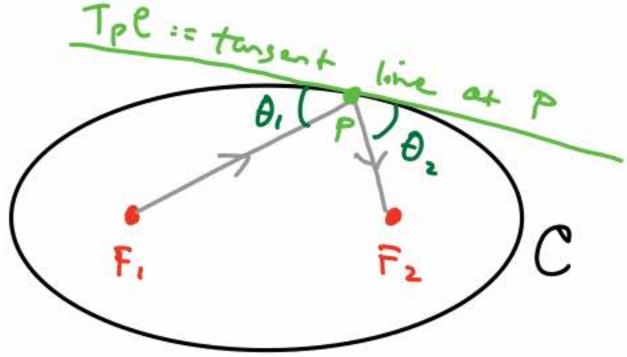
So (assuming $u'(s_0), F'(t_0) \neq 0$) the tangent line at a point on $\mathcal{C}$ only depends on $\mathcal{C}$ itself & not the parametrization.

(2) Note that if $X(t) = (t, f(t))$ is motion along the graph of $y = f(x)$, then (taking $t_0 = x_0$) $L(X(x_0), X'(x_0)) = L((x_0, f(x_0)), \underbrace{(1, f'(x_0))}_{\text{slope} = f'(x_0)})$ is the usual tangent line to the graph.

<u>Examples</u>:

- the tangent line to a line is the line itself (one param. of $L(P, A)$ is $Y(s) = P + sA$; the most general param. is $X(t) := Y(f(s))$)
- circular motion (in V_2): $Y(\theta) = (r\cos\theta, r\sin\theta) \rightsquigarrow Y'(\theta) = (-r\sin\theta, r\cos\theta) \rightsquigarrow Y''(\theta) = -Y(\theta)$, $\|Y'(\theta)\| = r$. For an arbitrary reparametrization, $X(t) = Y(\theta(t))$, have $X'(t) = \theta'(t) Y'(\theta(t))$ [note that $X' \cdot X = 0$!], with $|\theta'(t)|$ called the angular speed. If this is constant ($= \omega$), we get $X(t) = Y(\omega t)$, $X'(t) = \omega Y'(\omega t)$, $\|X'(t)\| = \omega r$, $X''(t) = \omega^2 Y''(\omega t) = -\omega^2 Y(\omega t)$ [the acceleration is centripetal, toward the center of the circle].
- elliptical motion (in V_2): $Y(\theta) = (a\cos\theta, b\sin\theta) \rightsquigarrow Y''(\theta) = -Y$ still true! (but speed $= (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2} \neq$ cst.)
- helical motion (in V_3): $Y(\theta) = (a\cos\theta, a\sin\theta, b\theta)$, $X(t) = Y(\omega t)$ $\rightsquigarrow X'(t) = \omega(-a\sin\omega t, a\cos\omega t, b) \rightsquigarrow X''(t) = \omega^2(-a\cos\omega t, -a\sin\omega t, 0)$.

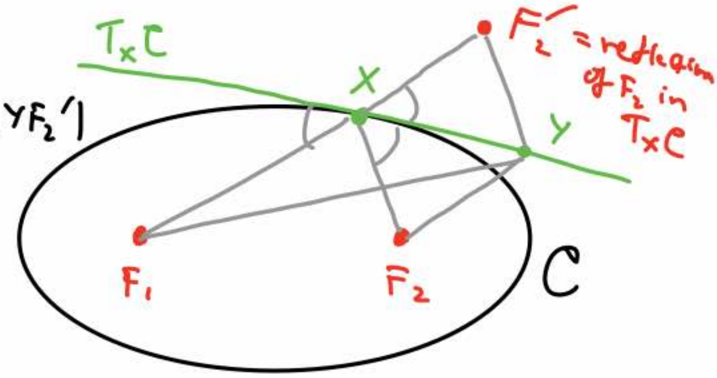
Elliptical billiards



On an elliptically-shaped pool table, suppose you shoot a ball from the focal point F_1. <u>Claim</u>: Ignoring friction/etc., the ball will bounce off the boundary and go straight through F_2. Equivalently, the angles θ_1 & θ_2 made by F_1P and PF_2 against the tangent line to C at P are equal. We now prove this in two different ways.

<u>Proof #1</u> (using a $\boxed{\text{fact}}$ I'll put on HW#13) Let X vary on C, Y vary on $T_X C$:

$$\text{constant} = |F_1X| + |XF_2| \underset{\substack{\uparrow \\ \text{b/c requires} \\ \text{stretching the "string" more}}}{\leq} |F_1Y| + |YF_2|$$
$$= |F_1Y| + |YF_2'|$$



with equality only if Y is on C, i.e. $Y = X$. But to say $Y = X$ minimizes $|F_1Y| + |YF_2'|$ is to say that $Y = X$ is the point on $T_X C$ along the segment F_1F_2'. So then all angles shown are congruent.

<u>Proof #2</u> (using vector calculus) Again let $X(t)$ trace out C, and write $d_i(t) = \|X(t) - F_i\|$, $X(t) - F_i = d_i(t)\,\hat{u}_i(t)$ ($\hat{u}_i(t)$ = unit vector). So

$$(X - F_i)\cdot\hat{u}_i = d_i \implies d_i' = (X - \underbrace{F_i}_{\text{constant}})'\cdot\hat{u}_i + \underbrace{(X-F_i)}_{\left(\substack{\text{parallel to}\\ \hat{u}_i}\right)}\cdot\underbrace{\hat{u}_i'}_{\left(\substack{\perp\text{ to}\\ \hat{u}_i}\right)} = X'\cdot\hat{u}_i + 0$$
$$= X'\cdot\hat{u}_i.$$

By HW#13, $d_1 + d_2 = \text{cst.} \implies 0 = d_1' + d_2' = X'\cdot\hat{u}_1 + X'\cdot\hat{u}_2 \implies$ (writing $T = \frac{X'}{\|X'\|}$) $\quad T\cdot\hat{u}_1 = -T\cdot\hat{u}_2$

hence $\cos\theta_1 = \cos\theta_2$.