Lecture 45 : Arc-length

Let's begin with a bit of review of last time. The notation in Apostol for vector-valued functions keeps "evolving" — first $F(x)$, then $X(t)$, now $\vec{r}(t)$ (my "$\rightharpoonup$" = his boldface) when discussing curvilinear motion. I encourage the use of the latter:

- $\vec{r}(t)$ = position at time t
- $\vec{v}(t) := \vec{r}\,'(t)$ = velocity at time t , $v(t) := \|\vec{v}(t)\|$ = speed
- $\vec{a}(t) := \vec{r}\,''(t) = \vec{v}\,'(t)$ = acceleration.

Ex 1/ $\vec{r}(t) = (\underbrace{\alpha \cosh \omega t}_{x(t)}, \underbrace{\beta \sinh \omega t}_{y(t)})$ components of $\vec{r}(t)$ — traces out a hyperbola since

$$\frac{x(t)^2}{\alpha^2} - \frac{y(t)^2}{\beta^2} = \cosh^2 \omega t - \sinh^2 \omega t = 1$$

$\vec{a}(t) = \vec{r}\,''(t) = \omega^2 \vec{r}(t)$

is in the same direction as $\vec{r}$

— opposite to the scenario for elliptic motion $\vec{r}(t) = (\alpha \cos \omega t, \beta \sin \omega t)$. //

Ex 2/ $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$ traces out a helix. Its tangent line at $t = \pi/2$ is $L\left(\vec{r}(\tfrac{\pi}{2}), \vec{v}(\tfrac{\pi}{2})\right) = L\left((0,3,2\pi), (-3,0,4)\right)$ which is parametrized by $\vec{q}(s) := (-3s, 3, 2\pi + 4s)$. //

Problem: The velocity of a bug at time t is given by $\vec{v}(t) = (t, t^2, t^3)$. It starts flying from the point $(1,1,1)$ at $t=0$. Find (a) its position at time $t=1$ and (b) the tangent line (at $t=1$) to the curve $\mathcal{C}$ it traces out.

Definition: (i) When $\vec{r}'(t) \neq 0$, the unit tangent vector $T(t) := \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$ is defined.
Captures direction of motion

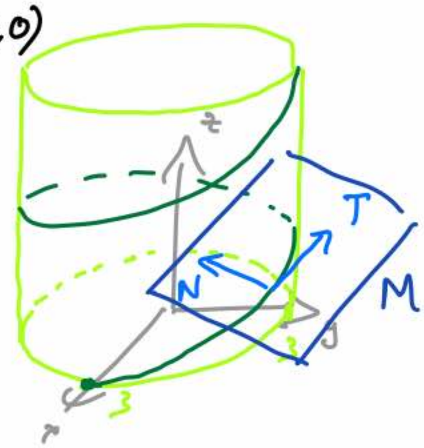
(ii) When $T'(t) \neq 0$, the unit normal vector $N(t) := \frac{T'(t)}{\|T'(t)\|}$ is defined.
Captures how T turns

(iii) If T, N are defined at t_0, the osculating plane to $\mathcal{C}$ at $\vec{r}(t_0)$ is given by $M(\vec{r}(t_0), \{T(t_0), N(t_0)\})$.

Ex 3/ $\vec{r}$ as in Ex 2 $\Rightarrow$ $\vec{r}' = (-3\sin t, 3\cos t, 4) \Rightarrow \|\vec{r}'\| = 5$ (constant!)

$\Rightarrow T = \frac{\vec{r}'}{\|\vec{r}'\|} = \left(-\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right)$, $T' = -\frac{3}{5}(\cos t, \sin t, 0)$ & $\|T'\| = 3/5$

$\Rightarrow N = \frac{T'}{\|T'\|} = (-\cos t, -\sin t, 0)$



Taking $t_0 = \pi/2$, $\vec{r}(t_0) = (0, 3, 2\pi)$, $T(t_0) = \left(-\frac{3}{5}, 0, \frac{4}{5}\right)$, $N(t_0) = (0, -1, 0)$, so the normal vector to M is $B := T(t_0) \times N(t_0) = \left(\frac{4}{5}, 0, \frac{3}{5}\right)$. Hence the equation of M is $(X - \vec{r}(t_0)) \cdot B = 0$ i.e. $\frac{4}{5}x + \frac{3}{5}(z - 2\pi) = 0$.

Note that since $T = \frac{\vec{r}'}{\|\vec{r}'\|} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{v}}{v}$, $\vec{v} = vT$.

Differentiating gives $\vec{a} = v'T + vT' = \underbrace{v'}T + \underbrace{v\|T'\|}N$, so
tangential & normal components of acceleration

both $\vec{v}$ & $\vec{a}$ lie in the osculating plane.

Now let $\vec{r} : [a,b] \to V_n$ trace out $\mathcal{C} = \vec{r}([a,b])$, and $P = \{t_0, t_1, \ldots, t_m\}$ $(a = t_0 < t_1 < \cdots < t_m = b)$ be a partition of $[a,b]$.

Let $\pi(P)$ denote the polygon with vertices $\{\vec{r}(t_k)\}$, with total length

$$|\pi(P)| := \sum_{k=1}^{m} \|\vec{r}(t_k) - \vec{r}(t_{k-1})\|.$$

Definition: (i) $\mathcal{C}$ is rectifiable $\Leftrightarrow$ $\mathcal{S} = \{|\pi(P)| \mid P \text{ partition of } [a,b]\}$ is bounded above.

(ii) If $\mathcal{C}$ is rectifiable, define its length ("$\ell(\mathcal{C})$" or "$\Lambda(a,b)$") by $\sup(\mathcal{S})$.

Theorem: If $\vec{v}(t)$ is continuous, then $\mathcal{C}$ is rectifiable and
$$\ell(\mathcal{C}) = \int_a^b v(t)\,dt .$$

Ex 4/ $\vec{r}(t) = (r\cos t, r\sin t)$, $t \in [0,\Theta]$ $\Rightarrow$ arc of a circle.

$\Rightarrow \vec{v}(t) = (-r\sin t, r\cos t) \rightarrow v(t) = r$. So $\ell(\mathcal{C}) = r\Theta$. //

If we let b move (take $b=t$), the arclength from $\vec{r}(a)$ to $\vec{r}(t)$ is $s(t) := \int_a^t v(u)\,du$, and $s'(t) = v(t)$. $\mathcal{C}$ is said to be parametrized by arclength if $v(t) \equiv 1$. We can obtain this by writing $t(s)$ for the inverse function and replacing $\vec{r}(t)$ by $\vec{q}(s) := \vec{r}(t(s))$.

Ex 5/ $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$, $t \in [0, \pi/2] \Rightarrow v(t) = \|\vec{r}'\| = 5$ (constant)

$\Rightarrow \ell(\mathcal{C}) = \int_0^{\pi/2} 5\,dt = \frac{5\pi}{2}$. A parametrization by arclength is given by $(3\cos t/5, 3\sin t/5, \frac{4}{5}t)$. //

Ex 6/ $\vec{r}(t) = (t, f(t))$, $t \in [a,b]$, parametrizes the graph Γ of $y = f(x)$ for $x \in [a,b]$.

Since $\vec{r}' = (1, f'(t))$, $v = \|\vec{r}'\| = \sqrt{1+(f'(t))^2}$ and so
$$\ell(\Gamma) = \int_a^b \sqrt{1+(f'(t))^2}\,dt .$$
//

Problem: Find the arclength of the curve traced out by
$$\vec{r}(t) = (t, 3t^2, 6t^3) \quad \text{on} \quad [0,2].$$

Proof of Theorem: First, $|\pi(P)| = \sum_{k=1}^m \|\vec{r}(t_k) - \vec{r}(t_{k-1})\| \underset{\text{FTC}}{=} \sum_{k=1}^m \left\| \int_{t_{k-1}}^{t_k} \overset{\vec{v}}{\vec{r}'(t)}\,dt \right\|$

$\leq \sum_{k=1}^m \int_{t_{k-1}}^{t_k} \overset{v(t)}{\|\vec{v}(t)\|}\,dt = \int_a^b v(t)\,dt$ gives an upper bound on $\mathcal{S}$,

and $\sup(\mathcal{S}) \leq$ any upper bound. So $\mathcal{C}$ is rectifiable and

$\underset{\Lambda(a,b)}{\ell(\mathcal{C})} \leq \int_a^b v(t)\,dt$. It remains to prove the opposite inequality.

To do this, we first verify an "additivity property": if

$\mathcal{C}_1 = \vec{r}([a,c])$ and $\mathcal{C}_2 = \vec{r}([c,b])$, then $\ell(\mathcal{C}) = \ell(\mathcal{C}_1) + \ell(\mathcal{C}_2)$. [Proof:

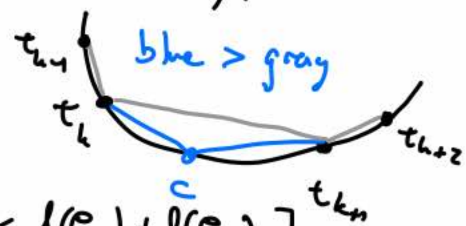
$\Lambda(a,b) = \Lambda(a,c) + \Lambda(c,b)$

if P_1, P_2 are partitions of $[a,c]$ & $[c,b]$, then taking $P = P_1 \cup P_2$ gives

$\ell(\mathcal{C}) \ge |\pi(P)| = |\pi(P_1)| + |\pi(P_2)| \Rightarrow \mathcal{C}_i$ rectifiable & $|\pi(P_1)| \le \ell(\mathcal{C}) - |\pi(P_2)|$

$\xRightarrow{\text{vary } P_1} \ell(\mathcal{C}_1) \le \ell(\mathcal{C}) - |\pi(P_2)| \Rightarrow |\pi(P_2)| \le \ell(\mathcal{C}) - \ell(\mathcal{C}_1) \xRightarrow{\text{vary } P_2}$

$\ell(\mathcal{C}_2) \le \ell(\mathcal{C}) - \ell(\mathcal{C}_1) \Rightarrow \ell(\mathcal{C}_1) + \ell(\mathcal{C}_2) \le \ell(\mathcal{C})$. Conversely, let P be arbitrary and let $P_c = P \cup \{c\} =: P_1 \cup P_2$. By the triangle inequality (see figure),



$|\pi(P)| \le |\pi(P_c)| = |\pi(P_1)| + |\pi(P_2)| \le \ell(C_1) + \ell(C_2) \Rightarrow \ell(\mathcal{C}) \le \ell(\mathcal{C}_1) + \ell(\mathcal{C}_2)$.]

Now let $s(t) := \begin{cases} 0, & t = a \\ \Lambda(a,t), & t > a \end{cases}$ be the arclength function,

which is increasing since $s(t_2) - s(t_1) = \Lambda(a,t_2) - \Lambda(a,t_1) = \Lambda(t_1,t_2) \ge 0$ using the additivity just proved. Define also $f(t) := \int_a^t v(u)\,du$, which has $f'(t) = v(t)$ [$\vec{v}$ cts. $\Rightarrow v$ cts.]. Then by the 1st ¶ of proof,

$\|\vec{r}(t+h) - \vec{r}(t)\| \le \Lambda(t,t+h) = s(t+h) - s(t) \le \int_t^{t+h} v(u)\,du$. Hence

$$\left\| \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right\| \le \frac{s(t+h) - s(t)}{h} \le \frac{1}{h}\int_t^{t+h} v(u)\,du = \frac{f(t+h) - f(t)}{h},$$

the end terms of which limit to $\|\vec{r}'(t)\| = v(t) = f'(t)$. By the Squeeze Theorem, we deduce the middle term's limit $s'(t)$ is also $v(t)$. $\square$