Lecture 46: Curvature

Though I'm going to start with...

Kepler's 2nd Law

We are looking at the motion (a priori in 3-space) of a planet in orbit around the sun, with position $\vec{r}(t)$, velocity $\vec{v} = \vec{r}\,'$, & acceleration $\vec{a} = \vec{r}\,''$. Newton postulates that

- $\vec{F} = m\vec{a}$ and $\vec{F} = \dfrac{-GMm}{r^3}\vec{r}$

(gravitational constant G, mass of sun M, mass of planet m)

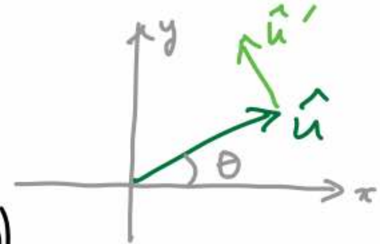
here $r = \|\vec{r}\|$, so $\dfrac{\vec{r}}{r^3} = \dfrac{1}{r^2}\dfrac{\vec{r}}{\|\vec{r}\|} = \dfrac{1}{r^2}\hat{u}$, with $\hat{u}$ a unit vector

from which we conclude that

$$\vec{a} = -\frac{GM}{r^3}\vec{r}\left(= -\frac{GM}{r^2}\hat{u}\right).$$

In particular, $\vec{a} \parallel \vec{r} \implies \vec{r}\times\vec{a} = \vec{0} \implies \dfrac{d}{dt}(\vec{r}\times\vec{v}) = \vec{v}\times\vec{v} + \vec{r}\times\vec{a} = \vec{0}$

$\implies \vec{r}\times\vec{v} =: \vec{C}$ is constant in t. Since also $\vec{r}(t) \perp \vec{C}$ $(\forall t)$, the motion is in a plane (the plane $\perp$ to $\vec{C}$). Choose coordinates so that $\vec{C} = C\hat{k}$.



Writing $\hat{u} = (\cos(\theta(t)), \sin(\theta(t)))$ since it has unit length, notice that $\hat{u}' = \theta'(t)\left(-\sin\theta(t), \cos(\theta(t))\right)$

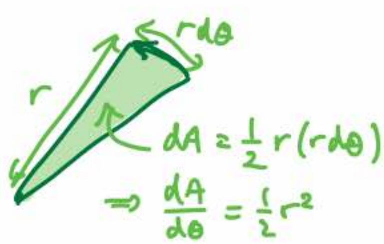
$\implies \|\hat{u}'\| = \dfrac{d\theta}{dt}$ (assuming the motion is counterclockwise). Moreover,

$\hat{u}' \perp \hat{u} \implies \|\hat{u}\times\hat{u}'\| = \underbrace{\|\hat{u}\|}_{1}\,\|\hat{u}'\|\,\underbrace{|\sin(\alpha_{\hat{u},\hat{u}'})|}_{1,\ \alpha = \pi/2} = \dfrac{d\theta}{dt} \implies$

$\vec{C} = \vec{r}\times\vec{r}\,' = r\hat{u}\times(r\hat{u})' = r^2\hat{u}\times\hat{u}' + rr'\underbrace{\cancel{\hat{u}\times\hat{u}}}_{0} = r^2\hat{u}\times\hat{u}'$

$\quad = r^2\|\hat{u}\times\hat{u}'\|\,\hat{k} = r^2\dfrac{d\theta}{dt}\hat{k}$

$\implies \dfrac{dA}{dt} = \dfrac{dA}{d\theta}\dfrac{d\theta}{dt} = \dfrac{1}{2}r^2\dfrac{d\theta}{dt} = \dfrac{1}{2}C$ is constant!



I'll at least prove Kepler's 1^{st} in the next class.

Turning to curvature, the idea is to have a measure, if one travels at unit speed along C, of how fast the direction (unit tangent vector T) is turning. Since it's hard to write down a unit speed parametrization (i.e. where $\Delta t = \Delta s$, s = arclength), we define the curvature vector by

$$\boxed{K := \frac{dT}{ds}} = \frac{dT/dt}{ds/dt} = \frac{T'(t)}{s'(t)} = \frac{T'(t)}{\underset{\text{speed}}{v(t)}} = \frac{\|T'(t)\|}{v(t)} \overset{\text{unit normal}}{N(t)}$$

and the curvature (of C at $\vec{r}(t)$) by

$$\boxed{\kappa(t) := \|K(t)\| = \frac{\|T'(t)\|}{v(t)}.}$$

If we consider this as a function on C (or of arclength) it does not depend on the choice of parametrization of C (why?).

Ex 1/ $\vec{r} = (R\cos t, R\sin t) \rightsquigarrow \vec{r}\,' = (-R\cos t, R\sin t)$, $v = R$ (const.)

$T = (-\sin t, \cos t)$, $T' = (-\cos t, -\sin t)$

so $\kappa = \frac{\|T'\|}{v} = \frac{1}{R}$. //

Motivated by this example, we define the radius of curvature of C at $\vec{r}(t)$ by $\boxed{\rho(t) := \frac{1}{\kappa(t)}}$. The circle centred at $\vec{r}(t_0) + \rho(t_0)N(t_0)$ with radius $\rho(t_0)$ is the osculating circle (to C) at $\vec{r}(t_0)$.

Ex 2/ $\vec{r} = (a\cos t, b\sin t) \rightsquigarrow \vec{v} = (-a\sin t, b\cos t)$, $\vec{a} = -\vec{r}$

$v = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$ varies between a & b

$T = \frac{r'}{v}$, $T' = \ldots$ uh oh. We need a formula for κ that doesn't force us to take derivatives of a $\sqrt{\ }$. //

We have $\vec{a} = \vec{v}' = (vT)' = v'T + vT' = v'T + v\|T'\|N$
$= v'T + \kappa v^2 N.$

If motion is in 3-space, we get

$$\underset{(\vec{r}'' \times \vec{r}')}{\vec{a} \times \vec{v}} = (v'T + \kappa v^2 N) \times (vT) = \kappa v^3 \underset{\text{unit vector}}{N \times T}$$

$$\Rightarrow \frac{\|\vec{a} \times \vec{v}\|}{v^3} = \kappa.$$ This avoids differentiating v.

For motion in a plane, $\vec{r}(t) = (x(t), y(t), 0)$ may be regarded as being in 3-space. So then $\vec{a} \times \vec{v} = (x''y' - x'y'')\hat{k} \Rightarrow$

$$\boxed{\kappa = \frac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{3/2}}}.$$

Ex 2, cont'd / We get

$$\kappa = \frac{|(-b\sin t)(-a\sin t) - (-a\cos t)(b\cos t)|}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} = \frac{ab}{v^3}$$

so $\rho = \frac{1}{\kappa}$ is between $\frac{\min(v^3)}{ab} = \frac{b^3}{ab} = \frac{b^2}{a}$ and $\frac{\max(v^3)}{ab} = \frac{a^2}{b}$.

Notice that if $a = b$ and our ellipse is a circle, $\frac{b^2}{a} = \frac{a^2}{b}$ is just its radius. From the figure, it should be apparent that the osculating circle is a better local approximation to $\mathcal{C}$ than the tangent line. //