Lecture 47: Kepler's 1st Law

Curvilinear motion in polar coordinates

Let $\vec{r}: I \to V_2$ be a vector-valued function (with continuous 1st & 2nd derivatives). Write $\vec{r}(t) = r(t)\left(\cos\theta(t), \sin\theta(t)\right) =: r(t)\,\tau_{\theta(t)}$, or $\boxed{\vec{r} = r\tau_\theta}$ for short. We have $\frac{d}{dt}\tau_{\theta(t)} = \frac{d}{dt}\left(\cos\theta(t), \sin\theta(t)\right) = \theta'(t)\left(-\sin\theta(t), \cos\theta(t)\right) = \theta'(t)\left(\cos\left(\theta(t)+\frac{\pi}{2}\right), \sin\left(\theta(t)+\frac{\pi}{2}\right)\right) = \theta'(t)\,\tau_{\theta(t)+\frac{\pi}{2}}$, or $\boxed{\tau_\theta' = \theta'\tau_{\theta+\frac{\pi}{2}}}$ for short. So then we have

- $\vec{v} = (r\tau_\theta)' = r'\tau_\theta + r\theta'\tau_{\theta+\frac{\pi}{2}}$

since $\tau_{\theta+\pi} = -\tau_\theta$

- $\vec{a} = \vec{v}' = r''\tau_\theta + r'\theta'\tau_{\theta+\frac{\pi}{2}} + r'\theta'\tau_{\theta+\frac{\pi}{2}} + r\theta''\tau_{\theta+\frac{\pi}{2}} - r(\theta')^2\tau_\theta$
  $= \left(r'' - r(\theta')^2\right)\tau_\theta + \left(r\theta'' + 2r'\theta'\right)\tau_{\theta+\frac{\pi}{2}}$
- $v = \|\vec{v}\| = \sqrt{(r')^2 + r^2(\theta')^2}$ since $\tau_\theta, \tau_{\theta+\frac{\pi}{2}}$ are orthogonal unit vectors

which can be used to calculate curvature as well.

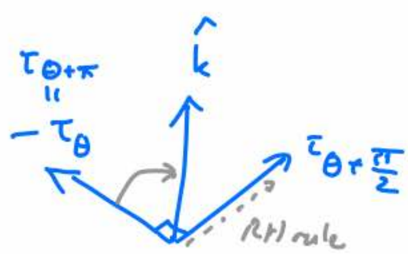
Proof of Kepler's 1st Law

See Apostol for proof of 3rd Law (easy but technical/not conceptual)

Recall where we left off in our proof of the 2nd law:

- $\vec{F} = m\vec{a}$ & $\vec{F} = -\frac{GMm}{r^3}\vec{r} \implies \vec{a} = -\frac{GM}{r^2}\frac{\vec{r}}{\|\vec{r}\|} = -\frac{GM}{r^2}\tau_\theta$.
- $\vec{r}\times\vec{a} = 0 \implies \vec{r}\times\vec{v} = \underbrace{\vec{C}}_{\text{constant}} = C\hat{k}$.

Now consider



$$\vec{a}\times\vec{C} = \left(-\frac{GM}{r^2}\tau_\theta\right)\times\left(r^2\theta'\hat{k}\right) = GM\theta'\tau_{\theta+\frac{\pi}{2}}.$$

Integrating both sides gives $\vec{v}\times\vec{C} = GM\tau_\theta + \underbrace{\vec{b}}_{} = GM(\tau_\theta + \vec{e})$,

constant of integration

and dotting with $\vec{r}$ gives

$$GM(\tau_\theta+\vec{e})\cdot\underbrace{\vec{r}}_{r\tau_\theta} = \vec{r}\cdot(\vec{v}\times\vec{C}) = \vec{C}\cdot(\vec{r}\times\vec{v}) = \vec{C}\cdot\vec{C} = C^2 .$$

Writing ϕ for the angle between $\vec{r}(t)$ and $\vec{e}$, and $e := \|\vec{e}\|$,

$$GMr(1+e\cos\phi) = C^2$$ becomes (with $d := \frac{C^2}{GMe}$)

$$\boxed{r(t) = \frac{ed}{1+e\cos\phi}}$$

which is the polar equation of a conic of eccentricity e and focus at the origin $\vec{O}$.

Planets are planets because they are in orbit (and not visitors from deep space which are flung back thereto), and so this conic must be an ellipse, i.e. $e\in(0,1)$.

For the remainder of the course we will be studying (real) vector spaces, or linear spaces in Apostol's terminology.

Definition: A (real) vector space is a set V together with two binary operations

$$+ : V\times V \longrightarrow V \quad \text{and} \quad \cdot : \mathbb{R}\times V \longrightarrow V$$

(vector addition: $(X,Y)\longmapsto X+Y$) (scalar mult.: $(r,X)\longmapsto rX$)

and "zero element" $O\in V$ such that $(\forall r,s,X,Y)$

- $X+Y = Y+X$
- $(X+Y)+Z = X+(Y+Z)$
- $r(sX) = (rs)X$
- $(r+s)X = rX+sX$
- $O+X = X$
- $1X = X$
- $X+(-1)X = O$ henceforth called "$-X$"
- $r(X+Y) = rX+rY$.

Remark: O is unique: if O' also satisfies this property, then

$$O' = O + O' = O' + O = O.$$

$-X$ is unique: if $X + Y = 0$, then adding $-X$ to both sides gives $-X + (X+Y) = -X + 0 \Rightarrow (-X+X) + Y = -X \Rightarrow Y = -X$.

$0X = O$: $\quad 0X + 0X = (0+0)X = 0X$

now add $-0X$ to both sides.

Ex 1/ Obvious example: V_n

Also the linear span $L(S)$ of a subset $S \subset V_n$

Or, the subset $W \subset V_n$ of vectors perpendicular to S //

Ex 2/ Less obvious: we can take V to be a set of functions:

- all real-valued functions on $[a,b]$
- all real-valued functions on $[a,b]$ with $f(a) = 0$ (why doesn't $f(a) = 1$ work?)
- all continuous real-valued fns. on $[a,b]$
- all differentiable real-valued fns. on $[a,b]$
- polynomials with real coefficients
- polynomials of degree $\leq d$ with real coeffts.
  (why don't polynomials of degree d work?)
- solutions of $y'' + ay' + by = 0$
  why not $F(x)$?

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Considering "vector spaces of functions" is not just the right way to study higher-order ODEs; it is how you break a sound wave or electrical signal into its constituent frequencies (Fourier analysis).

Problem: (1) Let $V := \mathbb{R} \times \mathbb{R}$, with operations $(x,y) + (z,w) = (x+z, y+w)$
$c(x,y) = (cx, y)$.

Is this a vector space?

(2) What if we replace the operations by

$$(x,y) + (z,w) = (x+z, 0) \quad \text{and} \quad c(x,y) = (cx, 0)\ ?$$