Lecture 48: More on abstract vector spaces

Subspaces

[N.B. I write $|\mathcal{S}|$ to denote the cardinality of a set, i.e. the number of elements it contains.]

Let $(V, +, \cdot, 0)$ be a (real) vector space, $W \subseteq V$ a subset.

<u>Definition</u>: W is a <u>subspace</u> if it is closed under $+$ & $\cdot$.

- Subspaces are vector spaces because the needed properties of $+$ & $\cdot$ are inherited from V, and closure under these operations means $(-1)w$ & $w+(-1)w = 0 \in W$.

Ex/ Given $\mathcal{S} \subset V$ subset, its linear span $L(\mathcal{S}) := \{\text{all finite sums } \sum_{v\in\mathcal{S}} c_v v\}$ is a subspace. Define $L(\emptyset) := \{0\}$. Different $\mathcal{S}$'s can generate the same subspace. //

Independence

<u>Definition</u>: $\mathcal{S}$ is <u>dependent</u> if there exists a <u>relation</u>, which means a finite sum $\sum_{v\in\mathcal{S}} c_v v = 0$ with not all $\{c_v\}$ zero, and <u>independent</u> otherwise. (For $|\mathcal{S}| < \infty$, this is equivalent to our earlier definition.)

- Subsets of independent sets are independent (b/c if a subset has a relation, the whole set does too by definition).

Ex/ Let V be the set of functions $f: \mathbb{R} \to \mathbb{R}$.

- $\mathcal{S} = \{\cos^2 x, \sin^2 x, 1\}$ is dependent.
- $\mathcal{S} = \{1, x, x^2, \dots, x^n\}$ is independent; its span $L(\mathcal{S})$ is denoted P_n.

  <u>Pf</u>: $\sum_{k=0}^{n} c_k x^k \equiv 0$ (the function is identically zero on $\mathbb{R}$) $\underset{\text{set } x=0}{\Longrightarrow}$ $c_0 = 0$ and the derivative $\sum_{k=1}^{n} k c_k x^{k-1} \equiv 0$ $\underset{x=0}{\Longrightarrow}$ $c_1 = 0$ etc. $\square$

- Given real #s $a_1 < \cdots < a_n$, $\mathcal{S} = \{e^{a_k x}\}_{k=1}^{n}$ is independent.

  <u>Pf</u>: induct on n: $\sum_{k=1}^{n} c_k e^{a_k x} \equiv 0 \Longrightarrow \sum_{k=1}^{n} c_k e^{(a_k - a_n)x} \equiv 0 \Longrightarrow$
  $c_n = \lim_{x\to\infty} \sum_{k=1}^{n} c_k e^{(a_k-a_n)x} = \lim_{x\to\infty} 0 = 0$ (so done by induction). $\square$ //

$\boxed{\text{Dimension}}$

Recall that in case $V = V_n$, we showed that

(*) for $|\mathcal{S}| = k$ & $\mathcal{S}$ independent, any $k+1$ elements of $L(\mathcal{S})$ are dependent.

[The same exact proof works for general V: to recap., if $\mathcal{S} = \{v_1, \dots, v_k\}$, we may write $w_1, \dots, w_{k+1} \in L(\mathcal{S})$ as $w_j = \sum c_{ij} v_i$ and represent this as a $k \times (k+1)$ matrix: $\begin{pmatrix} c_{11} & \cdots & c_{1,k+1} \\ c_{21} & \cdots & c_{2,k+1} \\ \vdots & \ddots & \vdots \\ c_{k1} & \cdots & c_{k,k+1} \end{pmatrix}$

we need to exhibit a relation on the "column vectors" for any such matrix. Arguing by induction, we are done if the entire 1^{st} row is zero (why?). Otherwise, use a nonzero entry in row 1 to eliminate all the other nonzero entries in row 1, by subtracting multiples of the column it's in from the other columns. We then get, by induction, a relation on the other columns.]

Definition: (i) If $V = L(\mathcal{S})$, V is spanned by $\mathcal{S}$. If V is spanned by a finite subset, V is called a finite-dimensional vector space.

(ii) A basis of a f.d. v.s. V is an independent set spanning V. Its dimension $\dim(V)$ is the number of elements in a basis.

Remark: Any 2 bases $\mathcal{S}$ & $\mathcal{T}$ of V have the same cardinality by (*): if $|\mathcal{T}| > |\mathcal{S}|$, then $\mathcal{T}$ would be dependent; & vice versa. Moreover, for a f.d. v.s., any independent set of cardinality $\dim(V)$ is a basis (Otherwise, adding a vector not in its span would violate (*)).

Ex/ $\dim(\{0\}) = 0$, $\dim(V_n) = n$, $\dim(P_n) = n+1$ (why?) //

Ex/ Suppose $r^2 + ar + b = 0$ has distinct real roots, and let $V = \{\text{solutions of } y'' + ay' + by = 0\}$. Then V is spanned by $e^{r_1 x}$ & $e^{r_2 x}$, so $\dim V = 2$. (Much the same story holds if V is the set of sequences solving a recurrence of the form $a_n = F(a_{n-1}, a_{n-2})$ $(n \geq 2)$. A basis is given by the solutions starting with $1, 0, \dots$ and $0, 1, \dots$.) //

- Given a basis $\mathcal{S} = \{v_1, \dots, v_n\}$, any $x \in V$ may be uniquely represented as $x = \sum_{i=1}^{n} c_i v_i$. We express this by writing $[x]_{\mathcal{S}} := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$.

Inner products

Definition: An inner product on V is a binary operation
$$(\, , \,) : V \times V \to \mathbb{R}$$
satisfying $(x,y) = (y,x)$, $(x, y+z) = (x,y) + (x,z)$, $(x, cy) = c(x,y)$, $(x,x) > 0$ if $x \neq 0$.
Write $\|x\| := \sqrt{(x,x)}$ for the norm. A vector space w/ inner product is an inner product space or (real) Euclidean space.

Ex/ $V = V_n$ has an inner product given by "$\cdot$" (the dot product).
But it has many other inner products, e.g. for $n=2$ we can take $((x_1,x_2),(y_1,y_2)) := 2x_1y_1 + x_1y_2 + x_2y_1 + y_1y_2$. //

Ex/ For $V = \underset{\substack{\text{continuous functions} \\ \text{on } [a,b]}}{C([a,b])}$ or P_n, $(f,g) := \int_a^b f(x)g(x)\,dx$ will work (why?) //

- By the same proofs as for the dot product, we get
  $|(x,y)| \leq \|x\| \, \|y\|$ (Cauchy–Schwarz) $\quad \left[\text{e.g. } \left(\int_a^b fg \, dx\right)^2 \leq \left(\int_a^b f^2 \, dx\right)\left(\int_a^b g^2 \, dx\right)\right]$
  $\|x+y\| \leq \|x\| + \|y\|$ (Δ inequality)
  $\|cx\| = |c| \, \|x\|$
- As for the dot product setting, we can define $\theta_{x,y} = \arccos\left(\frac{(x,y)}{\|x\| \, \|y\|}\right)$ and $x \perp y$ (x orthogonal to y) $\underset{\text{def.}}{\iff}$ $(x,y) = 0$. (Keep in mind that these depend on the choice of inner product!) A set S is orthogonal if all its elements are nonzero and mutually $\perp$, and orthonormal if its elements also have norm 1.
- Key point: any orthogonal set is independent (so if S is orthogonal with $|S| = \dim(V)$, then S is a basis of V).

  Proof: $\sum_i c_i x_i = 0 \underset{(x_j, \,)}{\implies} 0 = \sum_i c_i (x_j, x_i) = c_j \underbrace{(x_j, x_j)}_{\neq 0} \implies c_j = 0$. ☐
- Given an orthogonal basis $S = \{e_1, \ldots, e_n\}$ of V, any $v = \sum_i c_i e_i \underset{(e_j, \,)}{\implies}$ $(e_j, v) = \sum_i c_i (e_j, e_i) = c_j (e_j, e_j) \implies c_j = \frac{(e_j, v)}{(e_j, e_j)}$. If S is orthonormal (o.n.), this yields $v = \sum_{j=1}^n (e_j, v) e_j$ for every $v \in V$. Given $v, w \in V$, we get $(v, w) = \sum_{j=1}^n (e_j, v)(e_j, w)$.