Lecture 49: More on inner products

Recall that an inner product on a (real) vector space V is a "pairing"

$$(\, , \,) : V \times V \longrightarrow \mathbb{R}$$

[sending pairs of vectors $\vec{u}, \vec{v} \longmapsto (\vec{u}, \vec{v})$] which is

- symmetric: $(\vec{u}, \vec{v}) = (\vec{v}, \vec{u})$
- bilinear: $(\vec{u}, a\vec{v} + b\vec{w}) = a(\vec{u}, \vec{v}) + b(\vec{u}, \vec{w})$ and "vice versa"
- positive definite: $(\vec{u}, \vec{u}) \geq 0$ with equality $\Longleftrightarrow \vec{u} = \vec{0}$.

As for the dot product on V_n (which this generalizes), we have notions of

- length: $\|\vec{u}\| := (\vec{u}, \vec{u})^{1/2}$ [satisfies for instance $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$]
- distance: $\mathrm{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$
- orthogonality: $\vec{u} \perp \vec{v} \Longleftrightarrow (\vec{u}, \vec{v}) = 0$ [e.g., $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \Longleftrightarrow \vec{u} \perp \vec{v}$.
  Proof: $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v}, \vec{u}+\vec{v}) = (\vec{u}, \vec{u}) + (\vec{u}, \vec{v}) + (\vec{v}, \vec{u}) + (\vec{v}, \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2(\vec{u}, \vec{v})$.]

Ex/ I claimed last time that, on $\mathbb{R}^2 (= V_2)$, if we write $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$, then $(\vec{x}, \vec{y}) = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$ defines an inner product. It's clearly symmetric & bilinear. To see positive-definiteness, i.e. that $0 = (\vec{x}, \vec{x}) = 2x_1^2 + 2x_1x_2 + x_2^2 \Longrightarrow \vec{x} = 0$, we need to "change coordinates": $x_1 = u_1 + \varphi u_2$, $x_2 = -\varphi u_1 + u_2$, where $\varphi = \frac{1+\sqrt{5}}{2}$ has $\varphi^2 = \varphi + 1$.

Then $2x_1^2 + 2x_1x_2 + x_2^2 = 2(u_1+\varphi u_2)^2 + 2(u_1+\varphi u_2)(-\varphi u_1 + u_2) + (-\varphi u_1 + u_2)^2$

$$= (3-\varphi)u_1^2 + (3+4\varphi)u_2^2 \quad [u_1u_2 \text{ terms cancel out}]$$

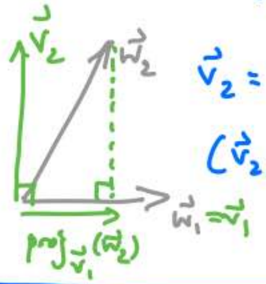
$u_1 = \frac{\tilde{u}_1}{\sqrt{3-\varphi}}, u_2 = \frac{\tilde{u}_2}{\sqrt{3+4\varphi}} \longrightarrow \; = \tilde{u}_1^2 + \tilde{u}_2^2$. Clearly this is only zero if $\tilde{u}_1 = 0 = \tilde{u}_2$, but then $x_1 = 0 = x_2$ (as desired)! Notice that in coordinates $\tilde{u}_1, \tilde{u}_2$, this weird inner product actually "becomes the dot product". That might suggest why it is natural to allow more general inner products. //

Gram-Schmidt Orthogonalization (Book is super long-winded here...)

Theorem: Let $W \subset V$ be a subspace with basis $\{\vec{w}_1, \dots, \vec{w}_k\}$.

Then
$$\vec{v}_1 = \vec{w}_1$$
$$\vec{v}_2 = \vec{w}_2 - \frac{(\vec{w}_2, \vec{v}_1)}{(\vec{v}_1, \vec{v}_1)} \vec{v}_1$$
$$\vec{v}_3 = \vec{w}_3 - \frac{(\vec{w}_3, \vec{v}_2)}{(\vec{v}_2, \vec{v}_2)} \vec{v}_2 - \frac{(\vec{w}_3, \vec{v}_1)}{(\vec{v}_1, \vec{v}_1)} \vec{v}_1$$
$$\vdots$$
produces an orthogonal basis of W.

Idea: $\text{proj}_{\vec{v}_1} \vec{w}_2 = \frac{(\vec{w}_2, \vec{v}_1)}{(\vec{v}_1, \vec{v}_1)} \vec{v}_1$ b/c



$\vec{v}_2 = \vec{w}_2 - \frac{(\vec{w}_2, \vec{v}_1)}{(\vec{v}_1, \vec{v}_1)} \vec{v}_1 \perp \vec{v}_1$:

$$(\vec{v}_2, \vec{v}_1) = (\vec{w}_2, \vec{v}_1) - \frac{(w_2, v_1)}{(v_1, v_1)}(v_1, v_1)$$
$$= (w_2, v_1) - (w_2, v_1) = 0.$$

Proof: Clearly $\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq W$. If we can show it is an orthogonal set, then it is also independent, and since it has k elements, a basis.

- To show the $\{\vec{v}_i\}$ are nonzero, just notice that the formulas exhibit the $\{\vec{w}_j\}$ as belonging to their linear span. If there are $< k$ nonzero $\vec{v}_i$'s, then the $\vec{w}_j$'s would be forced to be dependent, a contradiction.
- To show the $\{\vec{v}_i\}$ are $\perp$, let $A(m)$ be the assertion $\underbrace{\vec{v}_m \perp \vec{v}_1, \vec{v}_2, \dots, \vec{v}_{m-1}}$. $A(1)$ is true (vacuously). Assume $A(k-1)$. Then $\vec{v}_k = \vec{w}_k - \sum_{l=1}^{k-1} \frac{(\vec{w}_k, \vec{v}_l)}{(\vec{v}_l, \vec{v}_l)} \vec{v}_l$ has

$$(\vec{v}_k, \vec{v}_j) = (\vec{w}_k, \vec{v}_j) - \sum_{l=1}^{k-1} \frac{(\vec{w}_k, \vec{v}_l)}{(\vec{v}_l, \vec{v}_l)} \underbrace{(\vec{v}_l, \vec{v}_j)}_{\substack{=0 \text{ by induction} \\ \text{if } l \neq j}} = (\vec{w}_k, \vec{v}_j) - \underbrace{\frac{(\vec{w}_k, \vec{v}_j)}{\cancel{(\vec{v}_j, \vec{v}_j)}} \cancel{(\vec{v}_j, \vec{v}_j)}}_{l = j \text{ term}} = 0$$

$\Rightarrow A(k)$ holds. $\square$

Ex/ $V = C(-1,1)$, $(f,g) := \int_{-1}^{1} f(t) g(t)\, dt$, $W = L(\{\underset{\vec{w}_1}{1}, \underset{\vec{w}_2}{t}, \underset{\vec{w}_3}{t^2}\})$.

$$\vec{v}_1 = \vec{w}_1 = 1$$
$$\vec{v}_2 = \vec{w}_2 - \frac{(\vec{w}_2, \vec{v}_1)}{(\vec{v}_1, \vec{v}_1)} \vec{v}_1 = t - \frac{\cancelto{0}{\int_{-1}^{1} t \cdot 1\, dt}}{\int_{-1}^{1} 1 \cdot 1\, dt} 1 = t$$

first of the Legendre polynomials

$$\vec{v}_3 = \vec{w}_3 - \frac{(\vec{w}_3, \vec{v}_2)}{(\vec{v}_2, \vec{v}_2)} \vec{v}_2 - \frac{(\vec{w}_3, \vec{v}_1)}{(\vec{v}_1, \vec{v}_1)} \vec{v}_1 = t^2 - \frac{\cancelto{0}{\int_{-1}^{1} t^2 \cdot t\, dt}}{\int_{-1}^{1} t \cdot t\, dt} t - \frac{\int_{-1}^{1} t^2 \cdot 1\, dt}{\int_{-1}^{1} 1 \cdot 1\, dt} 1 = t^2 - \frac{1}{3}$$

For an o.n. (orthonormal) basis: write $\hat{v}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}$. Get
$$\hat{v}_1 = \frac{1}{\sqrt{2}}, \quad \hat{v}_2 = \sqrt{\frac{3}{2}}\, t, \quad \hat{v}_3 = \sqrt{\frac{45}{8}}\left(t^3 - \frac{1}{3}\right).$$

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Orthogonal Projections

(if $\{\vec w_1, \dots, \vec w_k\}$ is a basis of W)

Write $W^\perp := \{\vec v \in V \mid (\vec v, \vec w) = 0 \ \forall \vec w \in W\} = \{\vec v \in V \mid (\vec v, \vec w_j) = 0 \text{ for } j = 1, \dots, k\}$.

<u>Theorem</u>: If $\{\vec w_1, \dots, \vec w_k\}$ is an orthogonal basis of W, then

$$\operatorname{proj}_W \vec x := \sum_{l=1}^{k} \frac{(\vec w_l, \vec x)}{(\vec w_l, \vec w_l)} \vec w_l \quad \Big(= \sum_{l=1}^{k} (\vec w_l, \vec x) \vec w_l \text{ if o.n.} \Big) \text{ satisfies}$$

(i) $(\vec x - \operatorname{proj}_W \vec x) \in W^\perp$ and $\operatorname{proj}_W \vec x \in W$.

(ii) $\operatorname{proj}_W \vec x$ gives the "closest vector to $\vec x$" in W

(iii) $\|\vec x\| \geq \|\operatorname{proj}_W \vec x\|$ (projection decreases length), with equality $\Leftrightarrow \vec x \in W$.

<u>Proof</u>: (i) $$(\vec x - \operatorname{proj}_W \vec x, \vec w_i) = (\vec x, \vec w_i) - \sum_{l=1}^{k} \frac{(\vec w_l, \vec x)}{(\vec w_l, \vec w_l)} \underbrace{(\vec w_l, \vec w_i)}_{0 \text{ unless } l=i} = (\vec x, \vec w_i) - \frac{(\vec w_i, \vec x)}{\cancel{(\vec w_i, \vec w_i)}} \cancel{(\vec w_i, \vec w_i)} = 0.$$

(ii) $$\operatorname{dist}(\vec x, \vec w)^2 = \|\vec x - \vec w\|^2 = \|\underbrace{(\vec x - \operatorname{proj}_W \vec x)}_{\text{in } W^\perp} + \underbrace{(\operatorname{proj}_W \vec x - \vec w)}_{\text{in } W}\|^2 \underset{\Rightarrow \text{ orthogonal}}{=} \|\vec x - \operatorname{proj}_W \vec x\|^2 + \|\operatorname{proj}_W \vec x - \vec w\|^2$$

$$\geq \|\vec x - \operatorname{proj}_W \vec x\|^2 = \operatorname{dist}(\vec x, \operatorname{proj}_W \vec x)^2.$$

(iii) $$\|\vec x\|^2 = \|\underbrace{(\vec x - \operatorname{proj}_W \vec x) + \operatorname{proj}_W \vec x}_{\perp}\|^2 = \|\vec x - \operatorname{proj}_W \vec x\|^2 + \|\operatorname{proj}_W \vec x\|^2 \geq \|\operatorname{proj}_W \vec x\|^2. \qquad \square$$

Ex/ Given $f \in C[-1,1]$, with respect to the notion of distance conferred by $(f,g) = \int_{-1}^{1} fg \, dt$ (i.e. $\operatorname{dist}(f,g) = (f-g, f-g)^{1/2} = \left(\int_{-1}^{1} (f-g)^2 dt\right)^{1/2}$), by (ii) the "closest quadratic approximation to f" is $\operatorname{proj}_W f$. To compute this, use the o.n. basis $\hat v_1, \hat v_2, \hat v_3$ found in the last example; so if $f = \sin(\pi t)$, then

$$\operatorname{proj}_W(f) = \sum_{l=1}^{3} (\vec v_l, f) \vec v_l = \cancelto{0}{\left(\int_{-1}^{1} \frac{1}{\sqrt 2} \sin(\pi t) \, dt\right)} \frac{1}{\sqrt 2} + \left(\int_{-1}^{1} \sqrt{\tfrac{3}{2}}\, t \sin(\pi t) \, dt\right) \sqrt{\tfrac{3}{2}}\, t$$

$$+ \cancelto{0}{\left(\int_{-1}^{1} \sqrt{\tfrac{45}{8}} \left(t^2 - \tfrac{1}{3}\right) \sin(\pi t) \, dt\right)} \sqrt{\tfrac{45}{8}} \left(t^2 - \tfrac{1}{3}\right)$$

$$= \frac{3}{2} \left(\int_{-1}^{1} t \sin(\pi t) \, dt\right) t$$

$$= \frac{3}{\pi} t.$$

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