Lecture 5: Functions & Area

In previous courses you have probably seen a function described as a rule or machine that accepts a real number input and returns a real number output: $a \mapsto f(a)$. Maybe a is only allowed to be between -1 and $+1$, as with $f = \arccos$, or maybe it can be anything but 1, as with $g(x) = \frac{1-x^{n+1}}{1-x}$. We say that g has domain $\mathbb{R}\setminus\{1\}$, although we could also restrict to a smaller domain like $\{x \in \mathbb{R} \mid x > 1\}$. Or we might observe that on $\mathbb{R}\setminus\{1\}$,

$$\frac{1-x^{n+1}}{1-x} = \frac{(1-x)(1+x+\cdots+x^n)}{1-x} = \sum_{k=0}^{n} x^k,$$

and use this to extend g to a function on all of $\mathbb{R}$.

It is clearly fairly important to know which domain we are considering a function on, since this will affect the validity of assertions about it (like continuity), and so we will need to include this domain in its definition.

Let X and Y be sets. Their Cartesian product is the set of ordered pairs

$$X \times Y := \{(x,y) \mid x \in X,\ y \in Y\}.$$

Definition 1: A function from X to Y is a subset

$$f \subseteq X \times Y \quad \text{(usually written } f: X \to Y\text{)}$$

such that, for each $x \in X$, there is exactly one $y \in Y$ such that $(x,y) \in f$. This y is denoted $f(x)$; the set X is called the domain of f, Y is its codomain, and $f(X) := \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$ its range.

(Note that $(x,y) \in f \iff y = f(x)$; we shall use the latter notation. Also, I prefer to write the subset of $X \times Y$ as "Γ_f", and call it the graph of f. Here the Greek letter gamma (Γ) stands for "graph".)

Remark: Saying that there is exactly one $f(x)$ for each x means that each vertical line $\{x = x_0\} \subset X \times Y$ meets Γ_f exactly once. So functions are single-valued — when working with square roots, you need to choose positive ($\sqrt{\ }$) or negative ($-\sqrt{\ }$).

Ex/ A sequence of real numbers is really just a function $f: \mathbb{P} \to \mathbb{R}$ (or $\mathbb{P} \cup \{0\}$). The n^{th} term of the sequence is $f(n)$. Though we usually write this as "a_n" or similar. Standard examples include the Fibonacci sequence and $a_n = n!$

Ex/ Find $\Gamma_g \cap \Gamma_f$ if (a) $f(x) = x^3$ and $g(x) = x$

(b) $f(x) = x^2 - 2$ and $g(x) = 2x^2 + 4x + 1$ //

Ex/ Let $g(x) = \sqrt{4-x^2}$ on $X = [-2,2] := \{x \in \mathbb{R} \mid -2 \leq x \leq 2\}$.
Show $g\left(\frac{1}{t}\right) = \frac{\sqrt{4t^2-1}}{|t|}$ and $\frac{1}{2+g(x)} = \frac{2-g(x)}{x^2}$ (valid where?).
(Note that $\sqrt{a^2} = |a|$ for any $a \in \mathbb{R}$.) //

Ex/ HW #2 will have a problem about properties of polynomial fcns. Here is a sort of warm-up example: find all polynomials of degree ≤ 2 satisfying $p(3x) = p(x+3)$. //

Not all the functions we consider will have domain $X \subseteq \mathbb{R}$.

<u>Definition 2</u>: The set $\mathcal{M}$ of measurable plane regions is the minimal set of subsets of $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ such that:

(a) $\mathcal{M}$ contains the rectangles $[0, r] \times [0, s]$ for any $r, s \geq 0$;

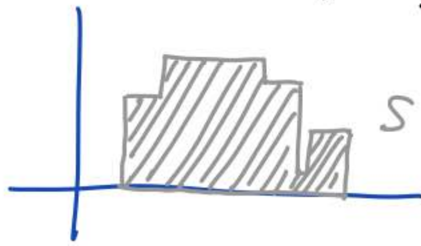
(b) $\mathcal{M}$ is closed† under rotations, translations, and reflections (i.e. congruence) ;

(c) $\mathcal{M}$ is closed under union, intersection, and difference ($A \cup B$, $A \cap B$, $A \setminus B$) ; and

† "closed" under an operation means that if you start with elements of the set & perform the operation, the output is still in the set.

(d) $\mathcal{M}$ contains the "well-approximable (ordinate) sets", defined as follows:

First, a step region $S \subset \mathbb{R}^2$ is a finite union of adjacent rectangles of the form $\bigcup_{i=1}^{m} [x_{i-1}, x_i] \times [0, y_i]$.



Write $\alpha(S) := \sum_{i=1}^{m} y_i (x_i - x_{i-1})$.

Then $Q \subset \mathbb{R}^2$ is well-approximable if there is exactly one number c such that $\alpha(S) \leq c \leq \alpha(T)$ for all step regions S, T with $S \subseteq Q \subseteq T$.

<u>Definition 3</u>: An area function (or Jordan measure) is a function

$$a : \mathcal{M} \longrightarrow \mathbb{R}^+ \cup \{0\} \ (=: \mathbb{R}_{\geq 0})$$

satisfying:

(i) $a([0, r] \times [0, s]) = rs$;

(ii) [for $A, B \in \mathcal{M}$] $a(A \cup B) = a(A) + a(B) - a(A \cap B)$;

(iii) [for $A \in \mathcal{M}$] if $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation, translation, or reflection, then $a(\tau(A)) = a(A)$; ("invariance under congruence")

[Note: (i)–(iii) ensure that $a(S) = \alpha(S)$ above for all step regions S.]

(iv) [for $A, B \in \mathcal{M}$ with $B \subseteq A$] $a(A \setminus B) = a(A) - a(B)$; and

(v) for $Q \in \mathcal{M}$ "well-approximable" as in (d), $a(Q) = c$. ("exhaustion property")

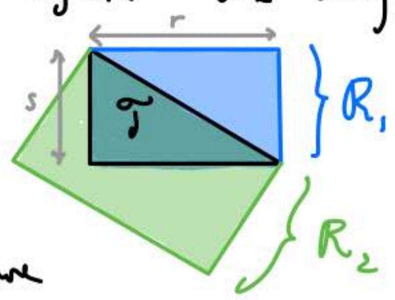
<u>Theorem</u>: There exists a unique area function.

<u>Proof</u>: (Can't make me do it! (Ha ha.) See Moise, the reference on Apostol p. 57: PDF may be found online. [The basic idea: define "a" first on polygonal regions by triangulating them (decompose into triangles) and adding up the triangles' areas; show that this area doesn't depend on the choice of triangulation. Then extend the area function to regions "well-approximable by polygons" (same idea as w/step functions), which is mostly formal: you have to verify, for example, that this doesn't conflict with the previous definition of area for polygons.] □

(Apostol takes this theorem as a postulate.)

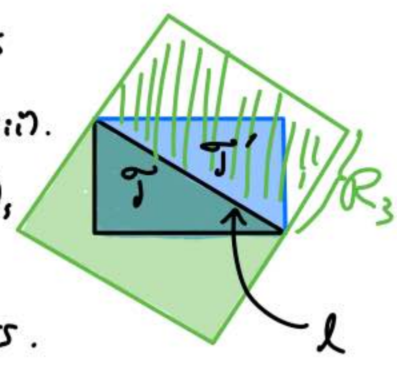
Ex/ By (a) & (b), all rectangles are in $\mathcal{M}$ (i.e. are measurable), and by (i) & (iii), if a rectangle R has side lengths r & s, $a(R) = rs$. A point is a rectangle with $r = s = 0$, and a line segment is a rectangle with (say) $s = 0$; clearly $a(\text{pt.}) = 0 = a(\text{segment})$. By (c) & (ii), the same goes for a finite union of points & line segments. //

Ex/ Let $\mathcal{T}$ be a right triangular region. We may view this as an intersection of rectangles:

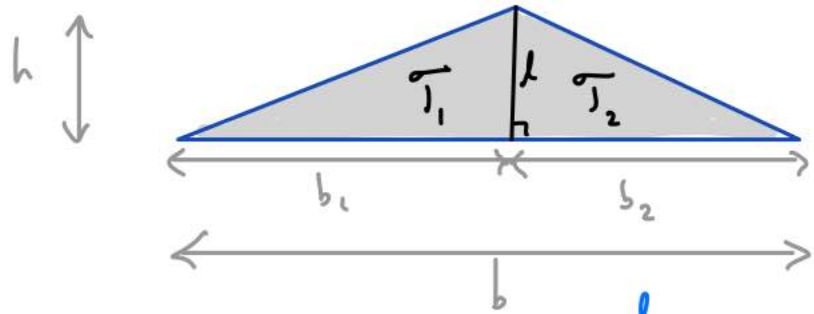


$\Rightarrow \mathcal{T} \in \mathcal{M}$.

But this doesn't tell us how to compute $a(\mathcal{T})$. We have to be more clever: the region $\mathcal{T}' = R_3 \cap R_1$ is a rotation of $\mathcal{T}$ hence has $a(\mathcal{T}') = a(\mathcal{T})$ by (iii). Moreover, $\mathcal{T} \cup \mathcal{T}' = R_1$ and $\mathcal{T} \cap \mathcal{T}' = \ell$ (segment), so (by the previous Example + (i)-(ii)) $rs = a(R_1) = a(\mathcal{T}) + a(\mathcal{T}') - \cancelto{0}{a(\ell)} = 2a(\mathcal{T}) \Rightarrow a(\mathcal{T}) = \frac{1}{2} rs$.



For a more general triangle Δ, just chop it up :



$$a(\Delta) = a(\sigma_1) + a(\sigma_2) - a(\overbrace{\sigma_1 \cap \sigma_2}^{\ell})$$

$$= \frac{1}{2} b_1 h + \frac{1}{2} b_2 h - 0 \qquad = \qquad \frac{1}{2} bh. \qquad //$$