Lecture 50: Linear Transformations

Nonlinear functions may predominate in STEM, but they are often well-approximated by linear ones. Linear/matrix algebra is the study of linear maps:

Definition 1: A linear transformation is a map (function) between two vector spaces

$$T : V \to W$$

obeying linearity: $T(\alpha \vec{v}_1 + \beta \vec{v}_2) = \alpha T(\vec{v}_1) + \beta T(\vec{v}_2)$. If $W = V$ then T is called an (linear) endomorphism of V.

Problem Which define endomorphisms of V_2?

$T(x,y) = (2x-y,\ x+y)$
$(x+1,\ y+1)$
$(x,\ 0)$
$(x^2,\ y^2)$
$(-x,\ y)$

$T(r,\theta) = (2r,\ \theta)$
$(r,\ 2\theta)$
$(r,\ \theta + \frac{\pi}{3})$

Geometrically describe the ones that do.

Ex/ Let V = vector space of all continuous functions $f: \mathbb{R} \to \mathbb{R}$.

Then $f(x) \longmapsto \int_0^x f(t)\,dt =: (Tf)(x)$ defines an endomorphism of V since $\int$ is linear: $\int_0^x (\alpha f_1(t) + \beta f_2(t))\,dt = \alpha \int_0^x f_1(t)\,dt + \beta \int_0^x f_2(t)\,dt$.

Alternatively, fixing $a, b \in \mathbb{R}$ we get a linear transformation $I: V \to \mathbb{R}$ by sending $f(x) \mapsto \int_a^b f(x)\,dx$. //

Definition 2: The image (or range) of T consists of all the vectors in W that are "hit" by the transformation:

$$\operatorname{im}(T) \text{ or } T(V) := \{\vec{w} \in W \mid \vec{w} = T\vec{v} \text{ for some } \vec{v} \in V\}.$$

The kernel (or null space) of T is the set of vectors in V "killed" by T:

$$\ker(T) \text{ or } N(T) := \{\vec{v} \in V \mid T\vec{v} = \vec{0}\}.$$

Proposition: $\ker(T) \subseteq V$ and $\operatorname{im}(T) \subseteq W$ are subspaces.

Proof: Recall that to show a subset is a subspace, it suffices to check that it contains all linear combinations of its own elements.

If $\vec{w}, \vec{w}' \in \operatorname{im}(T)$, then $\begin{cases} \vec{w} = T(\vec{v}) \\ \vec{w}' = T(\vec{v}') \end{cases}$ for some, $\vec{v}, \vec{v}' \in V$.

By linearity, $T(a\vec{v} + b\vec{v}') = aT(\vec{v}) + bT(\vec{v}') = a\vec{w} + b\vec{w}' \implies$ $a\vec{w} + b\vec{w}' \in \operatorname{im}(T)$.

If $\vec{v}, \vec{v}' \in \ker(T)$, then $T(\vec{v}) = \vec{0} = T(\vec{v}') \implies$

$T(a\vec{v} + b\vec{v}') = aT(\vec{v}) + bT(\vec{v}') = \vec{0} + \vec{0} = \vec{0} \implies a\vec{v} + b\vec{v}' \in \ker(T)$. $\square$

Ex/ (1) $I_V : V \to V$, $\vec{v} \mapsto \vec{v}$ is the identity transformation

$O_V : V \to V$, $\vec{v} \mapsto \vec{0}$ is the zero transformation

More generally, consider $cI_V : V \to V$, $\vec{v} \mapsto c\vec{v}$ (multiplication by c).

The kernel is zero unless $c = 0$.

(2) If V is endowed with an inner product $(,)$,

we can define

$$(\cdot, y) : V \longrightarrow \mathbb{R}$$
$$\vec{v} \longmapsto (\vec{v}, \vec{y}) \qquad (\text{kernel} = \text{vectors} \perp \text{to } \vec{y})$$

$$\text{proj}_W : V \longrightarrow W \qquad (\text{finite dim'l w/ o.n. basis } \hat{w}_1, \dots, \hat{w}_k)$$
$$\vec{v} \longmapsto \sum_{i=1}^{k} (\vec{v}, \vec{w}_i)\, \vec{w}_i \qquad (\text{kernel} = W^{\perp})$$

(3) Operations on polynomials

$$D : P_n \to P_n \quad (\text{or } P_{n-1})$$
$$f(x) \longmapsto f'(x) \qquad \text{differentiation } (\text{kernel} = \text{constants})$$

$$ev_{a_1, \dots, a_m} : P_n \longrightarrow V_m \qquad \text{evaluation map}$$
$$f(x) \longmapsto (f(a_1), \dots, f(a_m)) \qquad (\text{kernel} = \text{polynomials having all the } \{a_i\} \text{ as roots})$$

(4) $$T : \underbrace{C^2(a,b)}_{\text{fcns. on } (a,b) \text{ with continuous 2nd derivative}} \longrightarrow C(a,b) \qquad (\text{kernel} = \text{Solutions to } y'' + Py' + Qy = 0)$$

$$f \longmapsto f'' + Pf' + Qf = 0$$

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Definition 3: rank$(T) := \dim(\text{im}(T))$

nullity$(T) := \dim(\ker(T))$

Theorem: $\text{rank}(T) + \text{nullity}(T) = \dim(V)$.

Proof: Let $n := \dim(V)$, $k := \text{nullity}(T)$. (Of course, $n \geq k$.)

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be any basis for $\ker(T)$. By choosing vectors not in their span we can complete it to a basis

$$\{\vec{v}_1, \dots, \vec{v}_k ; \vec{v}_{k+1}, \dots, \vec{v}_n\}$$

of V.

Take any $\vec{w} \in \mathrm{im}(T)$. Then for some $\vec{v} \in V$,

$$\vec{w} = T\vec{v} = T\left(\sum_{i=1}^{n} \alpha_i \vec{v}_i\right) = \sum_{i=1}^{n} \alpha_i T(\vec{v}_i) \underset{\uparrow}{=} \sum_{i=k+1}^{n} \alpha_i T(\vec{v}_i)$$

$$(T\vec{v}_i = 0,\ i = 1, \dots, k)$$

$\Longrightarrow \{T\vec{v}_{k+1}, \dots, T\vec{v}_n\}$ span $\mathrm{im}(T)$.

To show they are linearly independent, first notice that if $0 = \sum_{i=k+1}^{n} a_i T(\vec{v}_i) = T\left(\sum_{i=k+1}^{n} a_i \vec{v}_i\right)$ then $\sum_{i=k+1}^{n} a_i \vec{v}_i \in \ker(T)$ by definition. Since $\{\vec{v}_1, \dots, \vec{v}_k\}$ span $\ker(T)$, $\sum_{i=k+1}^{n} a_i \vec{v}_i = \sum_{i=1}^{k} b_i \vec{v}_i$ (for some $b_i \in \mathbb{R}$); that is,

$$\vec{0} = b_1 \vec{v}_1 + \dots + b_k \vec{v}_k + (-a_{k+1}) \vec{v}_{k+1} + \dots + (-a_n) \vec{v}_n .$$

Since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a linearly independent set, all the coefficients (a_i & b_i) must be zero. So we have shown that

$$0 = \sum_{i=k+1}^{n} a_i T\vec{v}_i \quad \Longrightarrow \quad a_i = 0 \quad (i = k+1, \dots, n),$$

hence $\{T\vec{v}_{k+1}, \dots, T\vec{v}_n\}$ is L.I. $\Rightarrow$ basis of $\mathrm{im}(T) \Longrightarrow$

$\dim(\mathrm{im}(T)) = n - k$. $\square$

<u>Corollary</u>: $\dim(T(V)) \leq \dim V$.

Ex/ (Linear transformations from matrices)

Let $A = m \times n$ matrix. Define a L.T. by

$$L_A : V_n \longrightarrow V_m$$
$$\vec{x} \longmapsto A\vec{x}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{pmatrix}$$

The kernel is the space of $x_1, \dots, x_n$ solving the linear system

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$

of m equations.

Problem Suppose such a system, with $m = 8$ equations & $n = 15$ unknowns, has exactly 10 independent solutions $\vec{x}$.

What then is the dimension of the space of vectors $\vec{b} \in \mathbb{R}^8$ for which

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \quad \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_n \end{cases}$$

is solvable?