Lecture 52: Isomorphisms & Inverses

Let $T : V \to W$ be a linear transformation.

[Note: Most of this will be re-done next semester in terms of matrices.]

Definition: (i) T is onto or surjective if $\mathrm{im}(T) = W$,

in which case we write $T : V \twoheadrightarrow W$

(ii) T is 1-to-1 or injective if $\vec{v} \neq \vec{v}' \Rightarrow T\vec{v} \neq T\vec{v}'$.

(i.e. $T\vec{v} = T\vec{v}' \Longrightarrow \vec{v} = \vec{v}'$)

In this case we write $T : V \hookrightarrow W$.

(iii) If T is 1-to-1 & onto, it is an isomorphism, which is written $T : V \xrightarrow{\cong} W$. Two vector spaces are called isomorphic ($V \cong W$) if there exists an isomorphism between them, in either direction (though we'll soon see the two are equivalent).

idea: they are essentially the same vector space, differing only in presentation. (see Prop. 3 below.)

Proposition 1: T is 1-to-1 $\iff \ker(T) = \{\vec{0}\}$.

Proof: ($\Longrightarrow$) is clear: only $\vec{0}$ can go to $\vec{0}$

($\Longleftarrow$) Suppose $\ker(T) = \{0\}$, and let $T\vec{v} = T\vec{v}'$.

By linearity, $\vec{0} = T\vec{v} - T\vec{v}' = T(\vec{v} - \vec{v}')$, so

$\vec{v} - \vec{v}' \in \ker(T) = \{\vec{0}\} \Rightarrow \vec{v} - \vec{v}' = \vec{0} \Rightarrow \vec{v} = \vec{v}'$. $\square$

Ex / $W \subseteq V$ subspace of inner-product space $\Rightarrow$

inclusion $W \hookrightarrow V$, projection $V \twoheadrightarrow W$. //

Proposition 2: Assume V is finite-dimensional. Then: A linear transformation $T: V \to W$ is determined by where it sends a basis. That is, if $\{\vec{v}_1, \ldots, \vec{v}_n\} \subset V$ is a basis, and $\vec{w}_1, \ldots, \vec{w}_n \in W$ (not necessarily distinct!), then there is exactly one T with $T\vec{v}_k = \vec{w}_k$ for $k = 1, \ldots, n$.

If $\{\vec{w}_1, \ldots, \vec{w}_n\}$ are independent, then T is 1-to-1.

If they span W, then T is onto. If they're a basis, T is an isomorphism.

Proof: Any element $v = \sum_{k=1}^{n} \alpha_k \vec{v}_k \in V$ would have to be sent to $(T\vec{v} =) \sum_{k=1}^{n} \alpha_k T\vec{v}_k$ by linearity; and since the α_k are unique this also gives a definition of T.

If $\{\vec{w}_1, \ldots, \vec{w}_n\}$ is L.I., then $\vec{v} \in \ker(T) \Rightarrow$

$$0 = T\vec{v} = T\left(\sum_{k=1}^{n} \alpha_k \vec{v}_k\right) = \sum_{k=1}^{n} \alpha_k T\vec{v}_k = \sum_{k=1}^{n} \alpha_k \vec{w}_k \Rightarrow \alpha_k = 0 \quad (k = 1, \ldots, n)$$

$\Rightarrow \vec{v} = 0$. So T is injective.

If any vector $\vec{w} \in W$ is in the linear span of $\vec{w}_1 = T\vec{v}_1, \ldots, \vec{w}_n = T\vec{v}_n$, then $\vec{w} = \sum \alpha_k T\vec{v}_k = T(\sum \alpha_k v_k) \in im(T)$. $\square$

Ex/ Let $T: V_3 \to P_2$ be the L.T. determined by $T(\hat{i}) = 1$, $T(\hat{j}) = t$, $T(\hat{k}) = t^2$. By the Prop., this is an isomorphism. //

Proposition 3: Assume V, W are finite-dimensional. Then

$$\dim V = \dim W \iff V \cong W.$$

Proof: ($\Leftarrow$) Suppose $T: V \to W$ (or vice-versa) is an "$\cong$".

Then $\ker(T) = \{\vec{0}\} \Rightarrow \text{nullity}(T) = 0 \underset{\text{R+N Theorem}}{\Longrightarrow} \dim(V) = \text{rank}(T)$

$= \dim(im(T))$

$im(T) = W \leadsto = \dim(W)$.

($\Longrightarrow$) Suppose $\dim V = n = \dim W$. Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ and $\{\vec{w}_1, \dots, \vec{w}_n\}$ be bases. The L.T. defined (via Prop 2) to send $\vec{v}_j \mapsto \vec{w}_j$ $(\forall j)$ is an isomorphism. $\square$

We may compose linear transformations (just as we would compose any functions): e.g.,

$$U \xrightarrow{R} V \xrightarrow{T} W$$

is written $T \circ R$ (or TR). This is linear b/c

$$TR(\alpha\vec{x}+\beta\vec{y}) = T(R(\alpha\vec{x}+\beta\vec{y})) = T(\alpha R\vec{x}+\beta R\vec{y}) = \alpha TR\vec{x} + \beta TR\vec{y}.$$

<u>Definition</u>: Given $T: V \to W$, $S: W \to V$ is

(i) a <u>right inverse</u> for T if $T \circ S = I_W$

(ii) a <u>left inverse</u> for T if $S \circ T = I_V$

(iii) an <u>inverse</u> of T if it is <u>both</u>: write $S = T^{-1}$.

(In this case, we say T is <u>invertible</u>.)

<u>Theorem</u>: $T: V \to W$ has ...

(i) a right inverse $\Longleftrightarrow$ T is onto

(ii) a left inverse $\Longleftrightarrow$ T is 1-to-1

(iii) an inverse $\Longleftrightarrow$ T is an isomorphism.

<u>Proof</u>: (i) ($\Rightarrow$): if $TS = I_W$, then any $\vec{w} = I_W(\vec{w}) = T(S\vec{w}) \in \operatorname{im}(T)$.

($\Leftarrow$): I prove this only in the finite dimensional case (otherwise it requires ∞-dimensional bases & the Axiom of Choice)

Let $\{\vec{w}_1, \dots, \vec{w}_m\}$ = basis of W and choose $\vec{v}_1, \dots, \vec{v}_m$ with $T\vec{v}_j = \vec{w}_j$ (since T is onto). Define S to send $\vec{w}_j \mapsto \vec{v}_j$.

(ii) ($\Longrightarrow$): If $ST = I_V$, then $T\vec{v} = \vec{0}$ $\Rightarrow$ $\vec{v} = I_V\vec{v} = S(T\vec{v}) = S(\vec{0})$
$= \vec{0}$.
$\Rightarrow \ker(T) = \{\vec{0}\}$.

($\Longleftarrow$): Same comment as above

Let $\{\vec{w}_1, \dots, \vec{w}_n\}$ = basis of $\mathrm{im}(T)$. (So $\vec{w}_j = T\vec{v}_j$ for some $\{\vec{v}_1, \dots, \vec{v}_n\} \subset V$.) Extend this to a basis $\{\vec{w}_1, \dots, \vec{w}_n; \vec{w}_{n+1}, \dots, \vec{w}_m\}$ of W. Define S to send $\vec{w}_1 \mapsto \vec{v}_1, \dots, \vec{w}_n \mapsto \vec{v}_n$; $\vec{w}_{n+1}$ thru $\vec{w}_m \mapsto \vec{0}$.

(iii) ($\Longleftarrow$): Any $\vec{w} \in W$ is T of some $\vec{v}_{\vec{w}} \in V$, since T is onto. Since T is 1-to-1, this $\vec{v}_{\vec{w}}$ is unique. So define $S\vec{w} := \vec{v}_{\vec{w}}$ for each $\vec{w}$.

works in general

Clearly $T(S\vec{w}) = T(\vec{v}_{\vec{w}}) = \vec{w}$ by definition, and clearly $S(T\vec{v}) = \vec{v}$ as well. So $S = T^{-1}$. $\square$

Ex/ Let $a_0, a_1, \dots, a_n \in \mathbb{R}$ be distinct, and consider the evaluation map

$$T : P_n \to V_{n+1}$$
$$f(t) \longmapsto (f(a_0), \dots, f(a_n)).$$

Is this invertible / an isomorphism?

Set $$f_i(t) := \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{t - a_k}{a_i - a_k}$$ (product of these as k runs from 0 to n skipping i)

Then $$f_i(a_j) = \prod_{k \neq i} \frac{a_j - a_k}{a_i - a_k} = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases}.$$

Defining $S(b_0, \dots, b_n) := \sum_{i=0}^{n} b_i f_i(t)$, we have

$$TS(b_0, \dots, b_n) = T\left(\sum_{i=0}^{n} b_i f_i(t)\right) = \sum_{i=0}^{n} b_i T f_i(t)$$
$$= \left(\sum b_i f_i(a_0), \sum b_i f_i(a_1), \dots, \sum b_i f_i(a_n)\right) = (b_0, b_1, \dots, b_n).$$

So $T \circ S = Id_{V_{n+1}} \Rightarrow S$ is a right inverse

$\Rightarrow T$ is onto. By Rank + Nullity,

$$n+1 = \dim P_n = \dim(\operatorname{im}(T)) + \dim(\ker(T))$$
$$= \dim(V_{n+1}) + \dim(\ker(T))$$
$$= n+1 + \dim(\ker(T))$$

$\Rightarrow \ker(T) = \{0\} \Rightarrow T$ is 1-to-1 $\Rightarrow T$ is an isomorphism.

Notice that what we've really done here is shown that the function $\sum_{i=0}^{n} b_i f_i(x)$ takes prescribed values $b_0, \dots, b_n$ at $a_0, \dots, a_n$. This is called Lagrange interpolation. //

We conclude with the following observation, which came up in the example just done:

<u>Proposition 4</u>: If V & W are of the same finite dimension n, then the following are <u>equivalent</u> for $T: V \to W$:

(a) T is onto

(b) T is 1-to-1

(c) T is an isomorphism

<u>Proof</u>: Use Rank + Nullity:

$$\dim(\operatorname{im}(T)) + \dim(\ker(T)) = n.$$

If (a) holds, $\dim(\operatorname{im} T) = \dim(W) = n \Rightarrow \ker = \{0\} \Rightarrow$ (b).

If (b) holds, $\dim(\ker(T)) = 0 \Rightarrow \dim(\operatorname{im}(T)) = n \Rightarrow \operatorname{im}(T) = W$.

Since (a)+(b) is equivalent to (c), done. $\square$