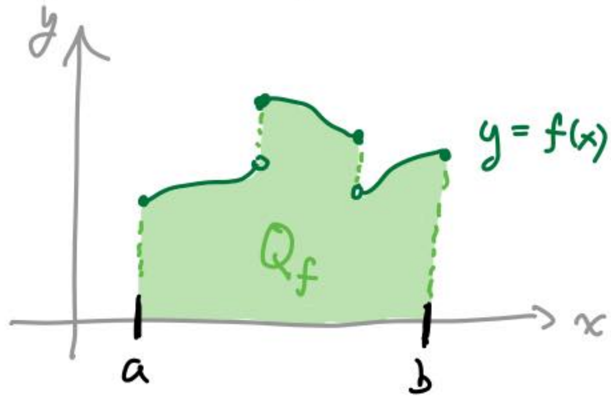
Lecture 7 : Integration of more general functions

We already have some intuition for how this should go. If the function f is ≥ 0 on $[a,b]$, its integral should be defined to be $a(Q_f)$, the area of the region "under the graph of f". But in some cases, we already know that this region Q_f has no "area" — either because



(1) f grows without bound somewhere and Q_f is too "big" (i.e., contains subsets with arbitrarily large area, like $f(x) = \begin{cases} 1/x & x > 0 \\ 0 & x = 0 \end{cases}$)

(2) f jumps up and down in a crazy way and we can't do a good job trapping Q_f between upper and lower step regions (like the $g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ example).

So we will eventually have to assume that f is "bounded" and "f doesn't oscillate too much" if we want an integral.

But let's try to make a general

Definition 1: Write

$$\mathcal{S} := \left\{ \int_a^b s(x)\,dx \;\middle|\; s \text{ step function}, \quad s \leq f \text{ on } [a,b] \right\}$$

$$\mathcal{T} := \left\{ \int_a^b t(x)\,dx \;\middle|\; t \text{ step function}, \quad f \leq t \text{ on } [a,b] \right\}$$

We say f is integrable on $[a,b]$ if there exists a unique $I \in \mathbb{R}$ with $s \leq I \leq t$ for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$. In this case, we write $I =: \int_a^b f(x)\,dx$ and call it the integral of f from a to b.

Definition 2: f is bounded on $[a,b]$ if there is an $M \in \mathbb{R}^+$ such that $-M \le f(x) \le M$ for all $x \in [a,b]$.

The first thing you may notice is that for f bounded, $\mathcal{S}$ & $\mathcal{T}$ are nonempty. (Why?) This is important!

Theorem 1: If f is bounded, there exists an $I \in \mathbb{R}$ with $\mathfrak{s} \le I \le \mathfrak{t}$ for all $\mathfrak{s} \in \mathcal{S}$, $\mathfrak{t} \in \mathcal{T}$. (But it may not be unique!)

Proof: For all s, t step functions with $s \le f \le t$ on $[a,b]$, we have $\mathfrak{s} = \int_a^b s(x)\,dx \le \int_a^b t(x)\,dx = \mathfrak{t}$. So any $\mathfrak{s} \in \mathcal{S}$ is a lower bound for $\mathcal{T}$ $\Rightarrow$ the "upper integral" $\overline{I}(f) := \inf \mathcal{T}$ (greatest lower bound) exists and $\mathfrak{s} \le \overline{I}(f)$. Hence $\overline{I}(f)$ is an UB for $\mathcal{S}$, and the "lower integral" $\underline{I}(f) := \sup \mathcal{S}$ exists and $\underline{I}(f) \le \overline{I}(f)$. Picking any $I \in [\underline{I}(f), \overline{I}(f)]$, we see that $I \le \overline{I}(f) \le \mathfrak{t}$ and $\mathfrak{s} \le \underline{I}(f) \le I$ for all $\mathfrak{s} \in \mathcal{S}$, $\mathfrak{t} \in \mathcal{T}$. $\square$

Boundedness solves issue (1). But we still have to deal with (2):

Definition 3: On any interval $\mathcal{I}$ (e.g. $[a,b]$, $[a,b)$, $(a,b]$, (a,b)), we say f is increasing (decreasing) if $x < y \Rightarrow f(x) \le f(y)$ ($f(x) \ge f(y)$). (It's called "strictly increasing" if $f(x) < f(y)$.) If f is either increasing or decreasing on $\mathcal{I}$, it is said to be monotonic there. Finally, f is piecewise monotonic on $[a,b]$ if there is a partition $P = \{\overset{a}{y_0}, y_1, \ldots, \overset{b}{y_m}\}$ and f is monotonic on each open interval (y_{j-1}, y_j).

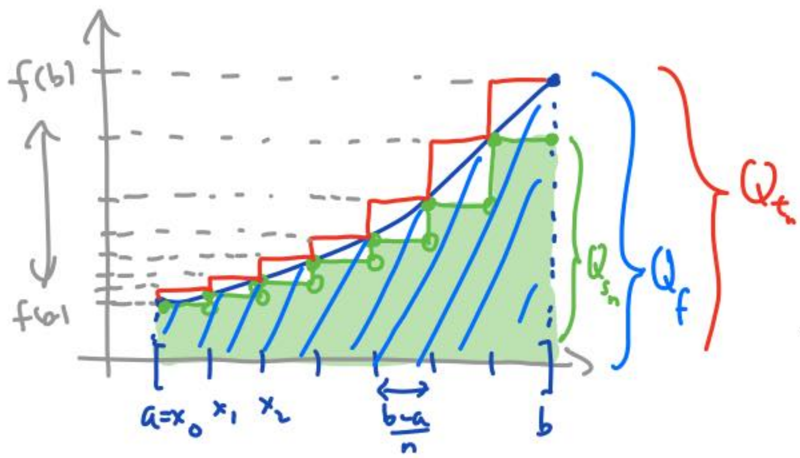
Theorem 2: f is integrable on $[a,b]$ if it is bounded and piecewise monotonic.

Proof: First, suppose f is increasing on $[a,b]$. Since f is bounded, we must show that $\underline{I}(f) = \overline{I}(f)$ to make I unique. Take the partition $P = \{x_0, x_1, \dots, x_n\}$ with $x_i = a + \frac{b-a}{n} i$. Consider the step functions

$$\begin{cases} s_n(x) := f(x_{i-1}) & \text{for } x \in [x_{i-1}, x_i) \\ t_n(x) = f(x_i) & \text{for } x \in [x_{i-1}, x_i) \end{cases} \quad \text{on } [a,b],$$

$]$ if $i=n$

which satisfy $s_n \le f \le t_n$: this means



$$s_n := \int_a^b s_n(x)\,dx = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1}) \in \mathcal{S}$$

and

$$t_n := \int_a^b t_n(x)\,dx = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) \in \mathcal{T}$$

so that

$$s_n \le \underline{I}(f) \le \overline{I}(f) \le t_n$$

$$\Rightarrow \quad 0 \le \overline{I}(f) - \underline{I}(f) \le t_n - s_n = \frac{b-a}{n}\left(\sum_{i=1}^{n}\left(f(x_i) - f(x_{i-1})\right)\right)$$

$$\stackrel{\text{telescoping sum!}}{=} \frac{b-a}{n}\left(f(x_n) - f(x_0)\right)$$

$$= \frac{b-a}{n}\left(f(b) - f(a)\right) = \frac{C}{n}$$

for every $n \in \mathbb{N}$

$\Rightarrow \quad \overline{I}(f) = \underline{I}(f)$, done.

this part of the proof is less important (and not so well written)

Next, if we change the values at a & b, this means that we have to modify $s_n(x)$ and $t_n(x)$ on $[x_0, x_1)$ and $(x_{n-1}, x_n]$, to (say) s_a, t_a, s_b, t_b. So $t_n - s_n$ becomes

$$\frac{b-a}{n}\left((t_a - s_a) + \sum_{i=2}^{n-1}\left(f(x_i) - f(x_{i-1})\right) + (t_b - s_b)\right) =$$

$$\frac{b-a}{n}\Big(\underbrace{t_a - s_a + t_b - s_b}_{K \text{ const.}} + \overbrace{f(x_{n-1})}^{\le M} - \overbrace{f(x_1)}^{\ge -M}\Big) \le \frac{b-a}{n}(K + 2M) = \frac{C}{n},$$

same story. So we only need f increasing on (a,b). Similarly, the case of f decreasing is dealt with.

Finally, if f is only piecewise monotonic, it is integrable on each closed subinterval $[y_{j-1}, y_j]$ of the partition, by the last 2 paragraphs. That is, the upper & lower integrals agree: $\overline{I}_j(f) = \underline{I}_j(f)$. But $\underline{I}(f)$ (for example) is the sup of sums of step integrals for the subintervals, which is the same as the sum of the sups, $\sum \underline{I}_j(f)$.
(Property II from Lecture 3)
Similarly, $\overline{I}(f) = \sum_j \overline{I}_j(f)$. So $\underline{I}(f) = \overline{I}(f)$. $\square$

You can see from the diagram in the proof that, if $f \geq 0$, we have $Q_{s_n} \subseteq Q_f \subseteq Q_{t_n}$, and that only one number — namely $I = \underline{I}(f) = \overline{I}(f) = \int_a^b f(x)\,dx$ — belongs to $[a(Q_{s_n}), a(Q_{t_n})] = [s_n, t_n]$ for all $n \in \mathbb{P}$. Hence $Q_f \in \mathcal{M}$ and $a(Q_f) = I$.

Properties (to be proved next week): for piecewise monotonic, bounded f, g:

① $\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$

② $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

③ $\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$

④ $\qquad = \frac{1}{k}\int_{ka}^{kb} f\left(\frac{x}{k}\right)\,dx$

Also: $\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx$ (for $a \leq b$)

⑤ $\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$ if $g \leq f$ on $[a,b]$.

⑥ [Key consequence of Theorem 2 & its proof] If f is $\left\{\begin{array}{l}\text{bounded}\\ \text{increasing}\end{array}\right.$ on $[a,b]$, then $\int_a^b f(x)\,dx$ is the (unique) number I satisfying $\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i) \leq I \leq \frac{b-a}{n}\sum_{i=1}^{n} f(x_i)$ for all $n \in \mathbb{P}$, where $x_i = a + \frac{b-a}{n} i$.

think: s_n think: t_n

Ex 1/ Let $f(x) = x^p$. You showed in HW#1 that

$$\underbrace{\frac{b}{n}\sum_{i=0}^{n-1}\Big(\underbrace{\frac{bi}{n}}_{x_i}\Big)^p}_{\text{lower sum}} \le \frac{b^{p+1}}{p+1} \le \underbrace{\frac{b}{n}\sum_{i=1}^{n}\Big(\underbrace{\frac{bi}{n}}_{x_i}\Big)^p}_{\text{upper sum}}$$

for every n. So $\int_0^b x^p\,dx = \frac{b^{p+1}}{p+1}$. More generally,

$$\int_a^b x^p\,dx \overset{②}{=} \int_a^0 x^p\,dx + \int_0^b x^p\,dx = \int_0^b x^p\,dx - \int_0^a x^p\,dx$$

$$= \frac{b^{p+1} - a^{p+1}}{p+1}.$$

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Ex 2/ $$\int_0^5 x^2(x-5)^4\,dx \overset{③}{=} \int_{-5}^0 (x+5)^2 x^4\,dx$$

$$= \int_{-5}^0 (x^6 + 10x^5 + 25x^4)\,dx$$

$$\overset{①}{=} \int_{-5}^0 x^6\,dx + 10\int_{-5}^0 x^5\,dx + 25\int_{-5}^0 x^4\,dx$$

$$\overset{\text{Ex 1}}{=} \frac{-(-5)^7}{7} + 10\,\frac{-(-5)^6}{6} + 25\,\frac{-(-5)^5}{5}$$

$$= 5^6/21.$$

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