Lecture 8: Properties of $\int_a^b f(x)\,dx$

Today we will prove the properties stated in the last lecture, thereby finishing Chapter 1 of Apostol. We will need three basic properties of sup/inf, the last two from Lecture 3:

(I) Given $\mathcal{A} \subseteq \mathcal{B}$ with $\mathcal{A}$ nonempty & $\mathcal{B}$ bounded above (below), $\sup \mathcal{A} \leq \sup \mathcal{B}$ ($\inf \geq \inf$). (Why?)

(II) Given $\mathcal{A}, \mathcal{B} \subset \mathbb{R}$ nonempty/bounded above (below), $\mathcal{A}+\mathcal{B} := \{a+b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ is too, and $\sup(\mathcal{A}+\mathcal{B}) = \sup(\mathcal{A}) + \sup(\mathcal{B})$ ($\inf$, $\inf$, $\inf$).

(III) Given $\mathcal{A}, \mathcal{B} \subset \mathbb{R}$ nonempty, with $a \leq b$ $\forall a \in \mathcal{A}, b \in \mathcal{B}$. Then $\sup \mathcal{A} \leq \inf \mathcal{B}$.

... and the version of the definition we will use:

A function f on $[a,b]$ is integrable if

$\underline{I}(f) := \sup\left\{ \int_a^b s(x)\,dx \;\middle|\; s \text{ step}, s \leq f \right\}$ and

$\overline{I}(f) := \inf\left\{ \int_a^b t(x)\,dx \;\middle|\; t \text{ step}, t \geq f \right\}$ exist and are equal,

and then $I(f) = \int_a^b f(x)\,dx := \underline{I}(f) = \overline{I}(f)$.

Now to the properties: f & g will be assumed integrable on $[a,b]$.

⑥ If f is bounded + increasing on $[a,b]$, then $I(f)$ is the (unique) number satisfying $s_n := \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i) \leq I(f) \leq \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) =: t_n$ for every $n \in \mathbb{P}$. (Here we take $x_i := a + \frac{b-a}{n} i$.)

Proof: Done in Lecture 7, but here's a different viewpoint: we have

$s_n \leq \underline{I}(f) \leq \overline{I}(f) \leq t_n \implies 0 \leq \overline{I}(f) - \underline{I}(f) \leq t_n - s_n = \frac{(b-a)(f(b)-f(a))}{n}$

for all $n \in \mathbb{P}$. By the Archimedean property, $\overline{I}(f) = \underline{I}(f)$. By the same token, any $I \in [s_n, t_n]$ $\forall n$ equals this common value. $\square$

(5) $\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx$ if $g \le f$ on $[a,b]$.

<u>Proof</u>: Since $s \le g \le f \le t \implies \underbrace{\int_a^b s(x)\,dx}_{(\alpha)} \le \underbrace{\int_a^b t(x)\,dx}_{(\beta)}$ by the comparison property of step integrals, we get by (III) that the $\{\sup$ of α's$\} \le \{\inf$ of β's$\} \implies \underline{I}(g) \le \overline{I}(f)$

$$\int_a^b g(x)\,dx \qquad \int_a^b f(x)\,dx \quad , \text{ done! } \square$$

(4) $\int_a^b f(x)\,dx = \frac{1}{k}\int_{ka}^{kb} f\left(\frac{x}{k}\right)dx \qquad (k \in \mathbb{R}^+)$

<u>Proof</u>: Set $F(x) := \frac{1}{k} f\left(\frac{x}{k}\right)$ on $[ka, kb]$. We have a 1-to-1 corresp. between step functions

$$\{ s \text{ on } [a,b] \mid s \le f \} \text{ and } \{ S \text{ on } [ka,kb] \mid S \le F \}$$

via $S(x) := \frac{1}{k} s\left(\frac{x}{k}\right)$ or $s(x) := k S(kx)$. Moreover, we had

$$\int_a^b s(x)\,dx = \int_{ka}^{kb} S(x)\,dx \quad \text{for step } S\text{'s},$$

and so $$\left\{ \int_a^b s(x)\,dx \;\middle|\; s \text{ step}, s \le f \right\} = \left\{ \int_{ka}^{kb} S(x)\,dx \;\middle|\; S \text{ step}, S \le F \right\}.$$

Taking sups gives $\underline{I}(f) = \underline{I}(F)$. Doing the same argument for $t \ge f$ etc. gives $\overline{I}(f) = \overline{I}(F)$. So $\underline{I}(F) = \overline{I}(F)$ (F is integrable) and $I(F) = I(f)$. $\square$

(1)(a) $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b (f(x) + g(x))\,dx$.

<u>Proof</u>: Write $h(x) := f(x) + g(x)$. We need to actually show h is integrable, like with F above. The key is to use (II) above: if $s_1 \le f$, $s_2 \le g$ then $s_1 + s_2 \le h$, and so

$$\left\{ \int_a^b s_1(x)\,dx \;\middle|\; s_1 \le f \text{ step} \right\} + \left\{ \int_a^b s_2(x)\,dx \;\middle|\; s_2 \le g \text{ step} \right\} =$$

(*) $$\left\{ \int_a^b (s_1+s_2)\,dx \;\middle|\; s_1 \le f, s_2 \le g \text{ step} \right\} \subseteq \left\{ \int_a^b s_3(x)\,dx \;\middle|\; s_3 \le h \text{ step} \right\}.$$

bounded above by any $\int t_1 + t_2$.

This set has (by (II)) supremum equal to the <u>sum</u> of supremums of the top two, i.e. $\underline{I}(f) + \underline{I}(g)$ — which since f & g are integrable is $I(f)+I(g)$. The inclusion in (*) insures $\sup(\text{LHS}) \leq \sup(\text{RHS})$, i.e. $I(f)+I(g) \leq \underline{I}(h)$. By working (mutatis mutandis) with $t_1 \geq f$, $t_2 \geq g$ etc., one has $\overline{I}(h) \leq I(f)+I(g)$.

Now since f & g are integrable, there exist step functions above & below f & g, which forces them — hence their sum h — to be bounded. That gives $\underline{I}(h) \leq \overline{I}(h)$. So altogether

$$\underline{I}(h) \leq \overline{I}(h) \leq I(f)+I(g) \leq \underline{I}(h)$$

$\Rightarrow$ $\overline{I}(h) = \underline{I}(h)$ (h is integrable) and $I(h) = I(f)+I(g)$. $\square$

1 (b) $$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx$$

2 $$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$$

3 $$\int_a^b f(x)\, dx = \int_{a+c}^{b+c} f(x-c)\, dx$$

Proofs of these last 3 properties may be found in Apostol, but 2 and 3 have simpler proofs when $f \geq 0$, by interpreting $\int_a^b f(x)\, dx$ as $a(Q_f)$ and using translation invariance / additivity of the area function.