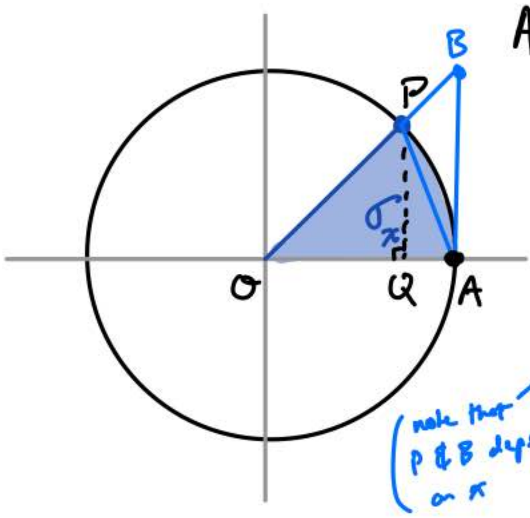
Lecture 9: Integrals of sin and cos

So far doing "integral calculus" before "differential calculus" seems a bit primitive — all we can integrate is polynomial & step functions! Let's fix that. Consider the unit circle $\{(X,Y) \mid X^2+Y^2=1\} \subset \mathbb{R}^2$.



Any circular sector (as shown) is measurable, because (i) the right triangle part is and (ii) the curvy part is the ordinate set of a decreasing (hence integrable) function. Write σ_x for the sector with $2a(\sigma_x) = x$. (Later, x will be the arclength, or radian measure of the "angle" at O, whatever that is. :))

(note that P & B depend on x)

π is defined to be the area of the unit disk shown

<u>Definition</u>: For $x \in [0, 2\pi]$, define $(\cos(x), \sin(x)) :=$ coords. of P.

Extend both functions to $\mathbb{R}$ by periodicity: for any $n \in \mathbb{Z}$, put

$\sin(x+2\pi n) := \sin(x)$, $\quad \cos(x+2\pi n) := \cos(x)$

<u>Properties which are immediate consequences of the definition:</u>

- $\sin$ (cos) is bounded and increasing (decreasing) (and ≥ 0) on $[0, \pi/2)$

  Actually $|\sin(x)| \leq 1$ (cos) and both functions are piecewise monotonic, hence integrable, on any closed interval.

- $\cos^2 x + \sin^2 x = 1$ (Pythagorean theorem for triangle OQP)

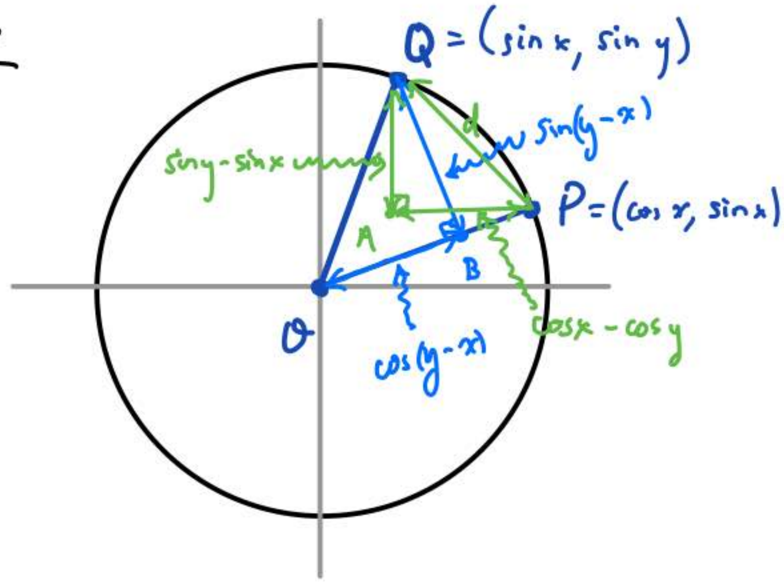
- $\cos(x - \frac{\pi}{2}) = \sin x$, $\quad \sin(x - \frac{\pi}{2}) = -\cos x$ (rotate figure by $90°$)

- $2a(\sigma_x) < 2a(OAB) \implies x < \frac{\sin x}{\cos x} \implies \boxed{\cos x < \frac{\sin x}{x}}$

- $2a(OAP) < 2a(\sigma_x) \implies \boxed{\sin(x) < x}$ ← both on $(0, \pi/2)$

- $\cos(-x) = \cos x$, $\quad \sin(-y) = -\sin(y)$

Properties which are consequences of the figure :



Pythagorean Theorem

for PAQ :

$$d^2 = (\sin y - \sin x)^2 + (\cos x - \cos y)^2$$
$$= \{\sin^2 y + \cos^2 y\} + \{\sin^2 x + \cos^2 x\} - 2\sin x \sin y - 2\cos x \cos y$$
$$= 2 - 2\sin x \sin y - 2\cos x \cos y$$

for PBQ :

$$d^2 = (1 - \cos(y-x))^2 + (\sin(y-x))^2$$
$$= 1 - 2\cos(y-x) + \{\cos^2(y-x) + \sin^2(y-x)\}$$
$$= 2 - 2\cos(y-x)$$

$$\Longrightarrow \boxed{\cos(y-x) = \sin x \sin y + \cos x \cos y}$$

Of course, this gives at once

$$\begin{cases} \cos(x+y) = \cos x \cos y - \sin x \sin y \\ \sin(x+y) = \cos\left(x+y-\frac{\pi}{2}\right) = \cos\left(x - \frac{\pi}{2}\right)\cos y - \sin\left(x - \frac{\pi}{2}\right)\sin(y) \\ \qquad\qquad = \sin x \cos y + \cos x \sin y \\ \sin(x-y) = \sin x \cos y - \cos x \sin y \end{cases}$$

$$\Longleftarrow \begin{cases} \sin 2x = 2\sin x \cos x \\ \left(\text{or } 2\sin x = \frac{\sin 2x}{\cos x}\right) \\ \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \end{cases}$$

$\vdots$

$$\Downarrow$$

$$\sin(x+y) - \sin(x-y) = 2\cos x \sin y$$

$$\Downarrow \quad x = \frac{a+b}{2},\ y = \frac{a-b}{2} \quad (\text{so } a = x+y,\ b = x-y)$$

These are enough to integrate $\cos(x)$ using

$$\boxed{\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right).}$$

$$(*) \begin{cases} \text{For } f \text{ decreasing on } [a,b],\ I(f) \text{ is the unique number satisfying} \\ \quad \frac{b-a}{n} \cdot \sum_{k=1}^{n} f(x_k) \leq I(f) \leq \frac{b-a}{n} \cdot \sum_{k=0}^{n-1} f(x_k) \quad (\forall n \in \mathbb{P}) \\ \text{where } x_k = a + \frac{b-a}{n}k. \end{cases}$$

Here's how : for $x \in [0, \pi/4n]$, using the properties above,

- $\sin(\underbrace{(2n+1)x}_{2nx+x}) - \sin(x) = \sin(2nx)\cos(x) + \underbrace{(\cos(2nx) - 1)}_{<0}\underbrace{\sin(x)}_{>0} < \sin(2nx)\frac{\sin(x)}{x}$

- $\sin((2n-1)x) + \sin x = \sin(2nx)\cos(x) + \underbrace{(1 - \cos(2nx))}_{2\sin^2 nx = \frac{\sin(nx)\sin(2nx)}{\cos(nx)}}\sin(x)$

$$\frac{\cos(nx)}{\cos((n-1)x)} = \frac{\cos((n-1)x)\cos x - \sin((n-1)x)\sin x}{\cos((n-1)x)}$$

$$= \cos x - \frac{\sin((n-1)x)\sin x}{\cos((n-1)x)} < \cos x < 1 < \frac{x}{\sin x}$$

$$= \frac{\sin(2nx)}{\cos(nx)}\big[\underbrace{\cos(nx)\cos(x) + \sin(nx)\sin(x)}_{\cos((n-1)x)}\big)$$

$$> \sin(2nx)\frac{\sin x}{x}$$

These two inequalities give, taking $x = \frac{a}{2n}$, $a \in (0, \frac{\pi}{2}]$

- $$\sin\left(\left(n+\tfrac{1}{2}\right)\frac{a}{n}\right) - \sin\left(\frac{a}{2n}\right) < \sin(a)\,\frac{\sin(a/2n)}{a/2n} < \sin\left(\left(n-\tfrac{1}{2}\right)\frac{a}{n}\right) - \sin\left(\frac{-a}{2n}\right)$$

telescoping sums

$$\sum_{k=1}^{n}\left\{\sin\left(\left(k+\tfrac{1}{2}\right)\frac{a}{n}\right) - \sin\left(\left(k-\tfrac{1}{2}\right)\frac{a}{n}\right)\right\} \qquad \sum_{k=0}^{n-1}\left\{\sin\left(\left(k+\tfrac{1}{2}\right)\frac{a}{n}\right) - \sin\left(\left(k-\tfrac{1}{2}\right)\frac{a}{n}\right)\right\}$$

use the $\sin a - \sin b$ identity

$$2\sin\left(\frac{a}{2n}\right)\sum_{k=1}^{n}\cos\left(\frac{ka}{n}\right) \qquad 2\sin\left(\frac{a}{2n}\right)\sum_{k=0}^{n-1}\cos\left(\frac{ka}{n}\right)$$

Multiplying thru by $\frac{a/2n}{\sin(a/2n)}$ finally gives (still with $a \in (0, \frac{\pi}{2}]$)

- $$\frac{a}{n}\sum_{k=1}^{n}\cos\left(\frac{ka}{n}\right) < \sin(a) < \frac{a}{n}\sum_{k=0}^{n-1}\cos\left(\frac{ka}{n}\right) \quad \text{for all } n \in \mathbb{P}$$

which by (*) tells us

$$\int_0^a \cos(x)\,dx = \sin(a).$$

- Apostol explains how to get this for all $a \in \mathbb{R}$, again using properties of cos & sin, then $\int_0^a \sin(x)\,dx = 1 - \cos(a)$.

(The argument for this is: $\int_0^a \sin(x)\,dx = \int_{-\pi/2}^{a-\pi/2} \underbrace{\sin\left(x+\frac{\pi}{2}\right)}_{\cos(x)}dx =$

$$\underbrace{\int_{-\pi/2}^{0}\cos(x)\,dx}_{= -\int_0^{-\pi/2}\cos(x)\,dx} + \int_0^{a-\frac{\pi}{2}}\cos(x)\,dx = -\sin\left(-\frac{\pi}{2}\right) + \sin\left(a - \frac{\pi}{2}\right)$$

$$= 1 - \cos(a).\ )$$

Finally, by the property $\int_a^b = \int_a^0 + \int_0^b = \int_0^b - \int_0^a$ of integrals generally, we conclude that

$$\int_a^b \cos(x)\,dx = \sin(b) - \sin(a) =: \sin(x)\Big|_a^b$$

$$\int_c^b \sin(x)\,dx = -\cos(b) + \cos(a) = -\cos(x)\Big|_a^b .$$

<u>Upshot</u>: You can <u>do</u> the more interesting integrals without antiderivatives & the Fundamental Theorem, but it might not be easy...