Lecture 34: Line integrals

A line (or path) integral is an integral over an oriented curve, which means a curve (or path) $C \subseteq \mathbb{R}^n$ with a choice of direction.

We will need C to be "piecewise smooth", for which my definition is a bit different from Apostol's.

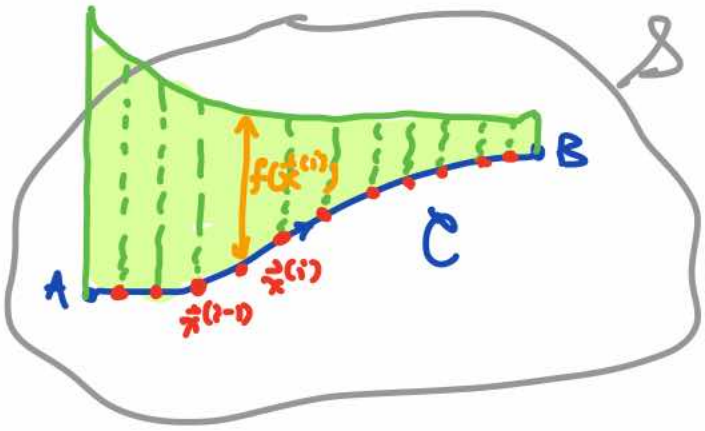
DEFINITION: A smooth parametrization of C is a 1-to-1, C^1 function $\vec{r}: [a,b] \to \mathbb{R}^n$ with image C, $\vec{r}(a) = A$ & $\vec{r}(b) = B$, and such that

$$\vec{r}\,'(t) \text{ is never } \vec{0} \text{ on } [a,b].$$

Apostol doesn't require

C is smooth if it has a smooth parametrization.

Let $f: \mathcal{S} \to \mathbb{R}$ be a piecewise continuous function whose domain $\mathcal{S} \subseteq \mathbb{R}^n$ contains C:



Think of f as having a graph "over" $\mathcal{C}$. How might we go about computing the area of the resulting (shaded) surface?

<u>Intuition</u>: f is a density function (e.g. charge, mass, etc.) on a wire, & we want the total (charge, mass, etc.).

<u>Practice</u>: partitioning $[a,b]$ by $t_i = a + \frac{b-a}{n} i$ and $\mathcal{C}$ by $\vec{x}^{(i)} := \vec{r}(t_i)$, we have

$$\mathcal{A} \approx \sum_{i=1}^{n} f(\vec{x}^{(i)})(\Delta s)_i \approx \sum_{i=1}^{n} f(\vec{r}(t_i))\,\|\vec{r}\,'(t_i)\|(\Delta t)_i ,$$

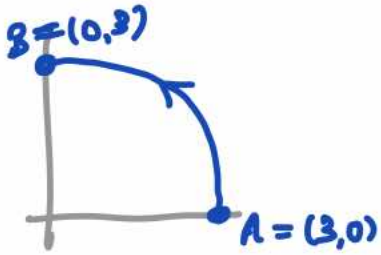
which in the limit gives $(\mathcal{A} =)$

$$(*) \qquad \boxed{\int_{\mathcal{C}} f\,ds \underset{\text{defn.}}{:=} \int_a^b f(\vec{r}(t))\underbrace{\|\vec{r}\,'(t)\|\,dt}_{\text{a.k.a. "}ds\text{"}}} ,$$

the <u>line integral of f with respect to arclength</u>.

Ex 1/ $\int_{\mathcal{C}} 1\,ds = \int_a^b \|\vec{r}\,'(t)\|\,ds$ is just the arclength! //

Ex 2/ Evaluate $\int_{\mathcal{C}} x^2 y\,ds$, where $\mathcal{C}$ is the quarter-circle



- $\vec{r}(t) = (3\cos t, 3\sin t)$ on $t \in [0, \pi/2]$

$$\int_{\mathcal{C}} x^2 y\,ds = \int_0^{\pi/2} 27\cos^2 t \sin t \sqrt{(-3\sin t)^2 + (3\cos t)^2}\,dt$$

$$= 81 \int_0^{\pi/2} \cos^2 t \sin t\,dt = 81 \underbrace{\left[-\tfrac{1}{3}\cos^3 t\right]_0^{\pi/2}}_{1/3}$$

$$= 27.$$

- $\vec{r}(t) = (\sqrt{9-t^2}, t)$ on $t \in [0,3]$

$$\int_{\mathcal{C}} x^2 y \, ds = \int_0^3 (9-t^2)\, t \underbrace{\sqrt{\left(\tfrac{1}{2}\frac{-2t}{\sqrt{9-t^2}}\right)^2 + 1^2}}_{\sqrt{\frac{t^2}{9-t^2}+1} = \sqrt{\frac{9}{9-t^2}}} \, dt$$

$$= \int_0^3 3t\sqrt{9-t^2}\, dt$$

$$= \left[-(9-t^2)^{3/2}\right]_0^3 = 9^{3/2} = 27.$$

The fact that the two parametrizations in Ex. 2 yield the same answer is a consequence of the fact that (*) can be described as a limit of Riemann sums with no reference to a specific parametrization.

$\vec{\mathcal{R}} : [c,d] \to \mathbb{R}^n$

Alternatively, if $\vec{r}(t)$ & $\vec{\mathcal{R}}(u)$ are two smooth parametrizations of $\mathcal{C}$, we can write $\vec{\mathcal{R}}(u) = \vec{r}(g(u))$ (with g C^1) and then

$g : [c,d] \to [a,b]$

$$\int_c^d f(\vec{\mathcal{R}}(u))\,\|\vec{\mathcal{R}}'(u)\|\, du = \int_c^d f(\vec{r}(g(u)))\,\|\vec{r}'(g(u))\|\,|g'(u)|\, du$$

remove since $g' > 0$

$t = g(u)$
$dt = g'(u)\,du$

$$= \int_a^b f(\vec{r}(t))\,\|\vec{r}'(t)\|\, dt$$

shows that (*) is independent of the choice of parametrization.

In fact, the orientation doesn't even matter because if g sends $[\to]$ to $[\leftarrow]$, the sign changes $\begin{cases} g'(u) \mapsto -g'(u) \\ \int_a^b \mapsto \int_b^a = -\int_a^b \end{cases}$

$c \quad d \qquad a \quad b$

cancel.

Ex 3/ Compute $\int_{\mathcal{C}} z\,ds$, where $\mathcal{C}$ is the line segment in $\mathbb{R}^3$ from $(3,0,1)$ to $(5,1,7)$.

Put $\vec{r}(t) = (1-t)(3,0,1) + t(5,1,7)$, $t \in [0,1]$.

Then $\int_{\mathcal{C}} z\,ds = \int_0^1 (1+6t)\sqrt{2^2+1^2+6^2}\,dt = \sqrt{41}\,[t+3t^2]_0^1$

$= 4\sqrt{41}$.

$z = (1-t)\cdot 1 + t\cdot 7 = 1+6t$

$\vec{r}'(t) = (5,1,7) - (3,0,1) = (2,1,6)$

//

Besides the line integral of $f(\vec{x})$ with respect to arclength, we can integrate with respect to one of the coordinates x_i:

$$(**) \quad \boxed{\int_{\mathcal{C}} f\,dx_i \underset{\text{defn}}{:=} \int_a^b f(\vec{r}(t))\,\underbrace{\vec{r}_i(t)\,dt}_{dx_i}.}$$

dx_i: $\vec{x} = \vec{r}(t)$ has i^{th} component $x_i = r_i(t)$

In 2 or 3 variables, these are written

$$\int_{\mathcal{C}} f\,dx, \quad \int_{\mathcal{C}} f\,dy, \quad \int_{\mathcal{C}} f\,dz.$$

Ex 4/ Evaluate the integral $\int_{\mathcal{C}} \underbrace{(x^2-y^2)\,dx + 2xy\,dy}_{\text{Combining them like this is typical notation}}$

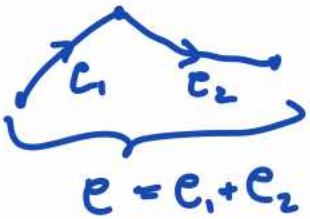
along the curve parametrized by $\vec{r}(t) = (t^2, t^3)$, $t \in [0,1]$.

- $dx = d(t^2) = 2t\,dt$, $dy = d(t^3) = 3t^2\,dt$
- so $\int_{\mathcal{C}} \cdots = \int_0^1 ((t^2)^2 - (t^3)^2)\,2t\,dt + 2(t^2)(t^3)\,3t^2\,dt$

$$= \int_0^1 \left(2t^5 - \cancel{2}t^7 + \overset{4}{\cancel{6}}t^7\right) dt$$

$$= \left[\tfrac{1}{3}t^6 + \tfrac{1}{2}t^8\right]_0^1 = \tfrac{1}{3} + \tfrac{1}{2} = \tfrac{5}{6}.$$

PROPERTIES OF (*) AND (**):

① 

$$\Rightarrow \int_C f \cdots = \int_{C_1} f \cdots + \int_{C_2} f \cdots$$

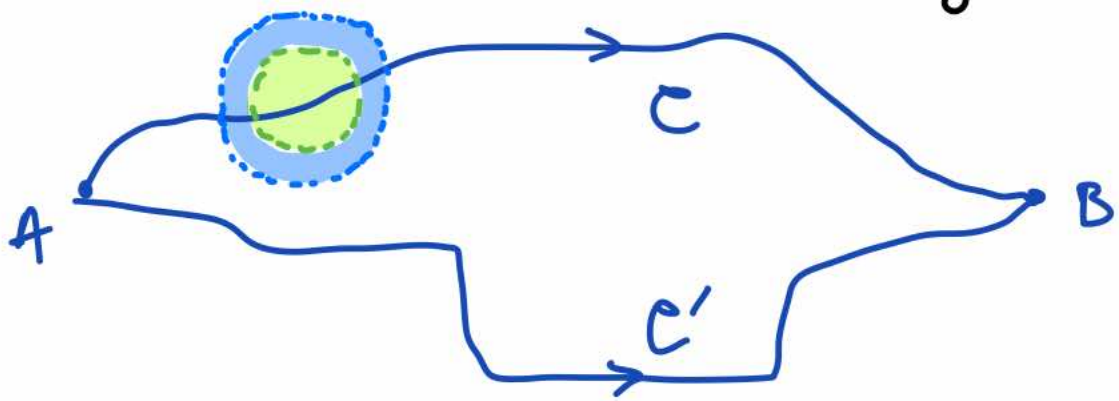
(for both (*) & (**))

② $\int_{-C} f\,ds = \int_C f\,ds$, but $\int_{-C} f\,dx_i = -\int_C f\,dx_i$

b/c $\sqrt{(x_1'(t))^2 + \cdots + (x_n'(t))^2}$ doesn't change sign, but $x_i'(t)$ does, when you reverse the path

③ independence of parametrization of a given oriented curve C

What about independence of path C from a given A to a given B?



Clearly this does <u>not</u> hold for (*) or (**) in general. Say $f \equiv 1$ on the green disk & $f \equiv 0$ outside the blue disk, and f is ≥ 0 and continuous everywhere. Then $\int_C f\,dx$, $\int_C f\,dy$, $\int_C f\,ds$

are all > 0 while $\int_{C'} f\,dx$, $\int_{C'} f\,dy$, $\int_{C'} f\,ds$ are all $= 0$.

We'll have to wait until next week to discover a situation where "independence of path" does hold.

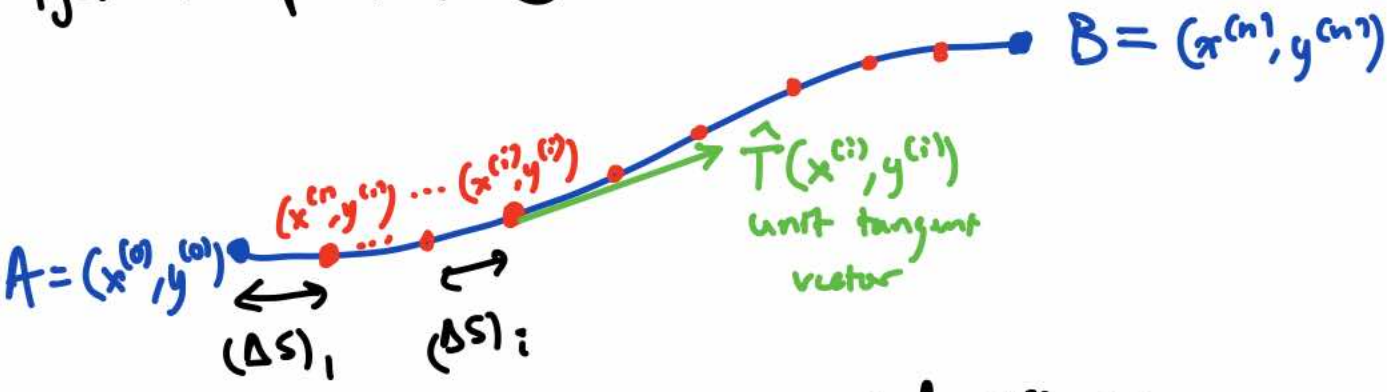
Now let's go back to the drawing board.

Suppose we have a particle moving in the plane along a curve C, in some sort of force field — assume continuous

$$\vec{F}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j} = (P(x,y), Q(x,y)).$$

What is the total work done by the particle?

Again we partition C



and estimate

$$(\Delta W)_i \approx \vec{F}(x^{(i)}, y^{(i)}) \cdot \underbrace{\hat{T}(x^{(i)}, y^{(i)})\,(\Delta s)_i}_{\text{approx. displacement vector along } i^{\underline{th}} \text{ subarc}}$$

$$W \approx \sum_{i=1}^{n} (\Delta W)_i$$

Taking the limit as the partition becomes finer and $n \to \infty$, we get

$$W = \int_{\mathcal{C}} \vec{F}(x,y) \cdot \hat{T}(x,y)\, ds$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \underbrace{\hat{T}(\vec{r}(t))}_{\vec{r}'(t)/\|\vec{r}'(t)\|} \|\vec{r}'(t)\|\, dt$$

parametrizing $\mathcal{C}$ by $\vec{r}(t)$, $t \in [a,b]$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

DEFINETION: Let $\vec{F} : \underset{\mathbb{R}^n}{\underset{\cap}{S}} \to \mathbb{R}^n$ be continuous, with S containing $\mathcal{C}$; and $\vec{r} : [a,b] \to \mathbb{R}$ a C^1 parametrization of $\mathcal{C}$. The line integral of $\vec{F}$ along $\mathcal{C}$ is

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

We can also write this as

(i) $\displaystyle\int_{\mathcal{C}} \vec{F} \cdot \hat{T}\, ds$ OR

$\vec{F} = (f_1, \dots, f_n)$

(ii) $\displaystyle\int_a^b \sum_{i=1}^n f_i(\vec{r}(t)) \underbrace{r_i'(t)\, dt}_{dx_i} = \int_{\mathcal{C}} f_1\, dx_1 + \dots + f_n\, dx_n$

from either of which one may deduce that

- $\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$ (why?)
- $\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_1+C_2} \vec{F}\cdot d\vec{r}$

not to mention

- $\int_C (a\vec{F}+b\vec{G})\cdot d\vec{r} = a\int_C \vec{F}\cdot d\vec{r} + b\int_C \vec{G}\cdot d\vec{r}$.

Independence of the choice of parametrization $\vec{r}$ of C follows from (i), and also by imitating the argument from before:

$$\begin{tikzcd} \overset{u}{[c,d]} \arrow[r, "g"] \arrow[rr, bend right, "\vec{\pi}"'] & \overset{t}{[a,b]} \arrow[r, "\vec{r}"] & \mathbb{R}^n \end{tikzcd}$$

$\vec{\pi}(u) = \vec{r}(g(u))$

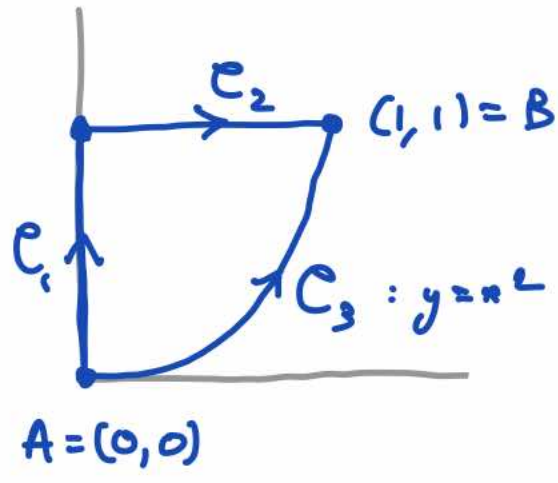
$$\int_c^d \vec{F}(\vec{\pi}(u))\cdot \vec{\pi}'(u)\,du = \int_c^d \vec{F}(\vec{r}(g(u)))\cdot \vec{r}'(g(u))\,g'(u)\,du$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)\,dt$$

$t = g(u)$
$dt = g'(u)\,du$

Ex 5/ Evaluate $\int_C \vec{F}\cdot d\vec{r}$ on the two paths from A to B indicated, where

$\vec{F}(x,y) = (xy^2, x^2-y^2)$.



$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_{C_3} xy^2\,dx + (x^2-y^2)\,dy$$

$\vec{r}(t)=(t,t^2)$

$$= \int_0^1 t(t^2)^2\,dt + (t^2-(t^2)^2)\overbrace{dt^2}^{2t\,dt}$$

$$= \int_0^1 (\cancel{t^5} + 2t^3 - \cancel{2}t^5)\,dt$$

$$= \left[\tfrac{1}{2}t^4 - \tfrac{1}{6}t^6\right]_0^1 = \tfrac{1}{3}.$$

$$\int_{C_1+C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

use $\vec{r}(t)=(0,t)$ $\Rightarrow d\vec{r}=(0,dt)$

use $\vec{r}(t)=(t,1)$ $\Rightarrow d\vec{r}=(dt,0)$

$$= \int_0^1 (0^2-t^2)\,dt + \int_0^1 t(1)^2\,dt$$

$$= -\frac{t^3}{3}\Big|_0^1 + \frac{t^2}{2}\Big|_0^1 = -\frac{1}{3}+\frac{1}{2} = \frac{1}{6}$$

So independence <u>of path</u> from A to B fails. //

APPLICATION:

If $\vec{r}(t)$ is the motion of an object in a force field $\vec{F}(\vec{x})$, then Newton's 2nd Law says

$$\vec{F}(\vec{r}(t)) = m\vec{r}''(t) = m\vec{v}'(t).$$

The work done by the force on the object

as it moves along $\mathcal{C}$ is therefore

$$W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b m \underbrace{\vec{v}\,'(t) \cdot \vec{v}(t)}_{\frac{1}{2}\frac{d}{dt}\left[\vec{v}(t)\cdot\vec{v}(t)\right]}\, dt$$

FTC

$$= \frac{1}{2} m \|\vec{v}(t)\|^2 \Big|_a^b$$

recall speed $v = \|\vec{v}\|$

$$= \frac{1}{2} m (v(b))^2 - \frac{1}{2} m (v(a))^2$$

$$= \Delta KE,$$

the change in the kinetic energy of the object.