

# Lecture 4: Parametrizing solutions

## Review of "Span"

Recall that the span of a collection of vectors is the set of all linear combinations of them:

$$\begin{aligned} \text{Ex 1/ } \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -1 \\ 0 \end{pmatrix} \right\} &= \left\{ s \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 3 \\ -1 \\ 0 \end{pmatrix} \mid s, t \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} s \\ 0 \\ -2s \\ -s \end{pmatrix} + \begin{pmatrix} -t \\ 3t \\ -t \\ 0 \end{pmatrix} \mid s, t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} s-t \\ 3t \\ -2s-t \\ -s \end{pmatrix} \mid s, t \in \mathbb{R} \right\} // \end{aligned}$$

So if you are asked

"Taking  $W = \text{set of all vectors of the form } \begin{pmatrix} 3s-4t \\ -4s+4t \\ -4s-4t \\ -3s+5t \end{pmatrix},$   
find  $\vec{u}$  &  $\vec{v}$  such that  $W = \text{span}\{\vec{u}, \vec{v}\}$ "

you'll just reverse the process, obtaining

$$\vec{u} = \begin{pmatrix} 3 \\ -4 \\ -4 \\ -3 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} -4 \\ 4 \\ -4 \\ 5 \end{pmatrix}.$$

# Homogeneous linear systems

These are systems of the form

$$A\vec{x} = \vec{0}.$$

They always have the "trivial solution"  $\vec{x} = \vec{0}$ , and the solution set is always the span of a set of vectors (as we now demonstrate).

Ex 2 / Determine all solutions of 
$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 0 \\ -4x_1 - 4x_2 - 8x_3 = 0 \\ -3x_2 - 3x_3 = 0. \end{cases}$$

$$\left[ \begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 \end{array} \right]$$

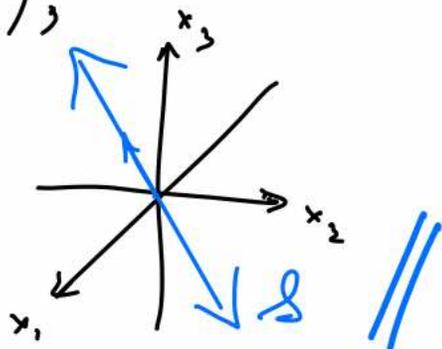
$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] = \text{rref}$$

$x_3$  free

[N.B.: You may drop the right-hand column of 0's if you are comfortable doing that.]

$$\text{So } \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

and the solution set is  $\mathcal{L} = \text{span} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$ .



## In homogeneous linear systems

These are the systems  $A\vec{x} = \vec{b}$  with  $\vec{b} \neq \vec{0}$ .

In this case,  $\vec{x} = \vec{0}$  is NEVER a solution. So the solution set  $\mathcal{S}$  can't be a span: if nonempty, it will be a "parallel translate" of a span.

Ex 3 / Determine all solutions of 
$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 8 \\ -4x_1 - 4x_2 - 8x_3 = -16 \\ -3x_2 - 3x_3 = 12 \end{cases}$$

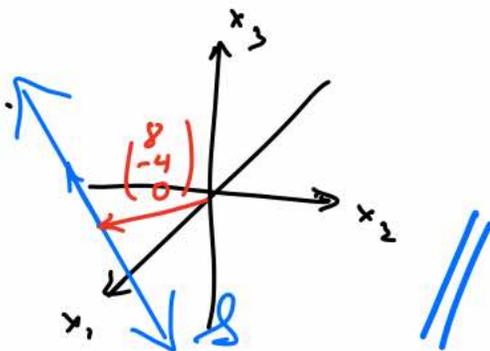
$$\left[ \begin{array}{ccc|c} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 12 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array} \right] = \text{ref}$$

$x_3$  free

$$\Rightarrow \begin{cases} x_1 = 8 - x_3 \\ x_2 = -4 - x_3 \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} 8 - x_3 \\ -4 - x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \mathcal{S} = \left\{ \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$



Here is another homogeneous system problem, with the matrix already nearly row-reduced:

Ex 4 / If  $A \sim$   $\begin{pmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ ,  
row-equiv.

describe the solution set to  $A\vec{x} = \vec{0}$ .

$$\rightarrow \begin{pmatrix} 1 & -4 & 0 & 0 & 3 & -7 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{ref}(A)$$

$\uparrow$   $\uparrow$   $\uparrow$   
 $x_2$   $x_4$   $x_6$  free

Now mentally adjoining a column of zeroes to the right,

we get  $\begin{cases} x_1 = 4x_2 - 5x_6 \\ x_3 = x_6 \\ x_5 = 4x_6 \end{cases} \Rightarrow$

$$\mathcal{A} = \left\{ \begin{pmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{pmatrix} \mid x_2, x_4, x_6 \in \mathbb{R} \right\}$$

$$= \left\{ x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix} \mid x_2, x_4, x_6 \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}$$



# Relation between homogeneous & inhomogeneous systems

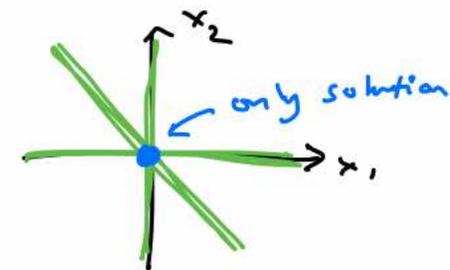
Consider  $A\vec{x} = \vec{0}$  and  $A\vec{x} = \vec{b} (\neq \vec{0})$ , with solution sets  $\mathcal{S}_0$  and  $\mathcal{S}_b$ .

Case I:  $\mathcal{S}_b$  empty

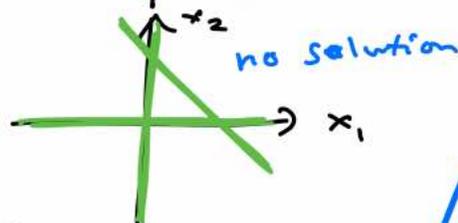
Ex 5 / 
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\Leftrightarrow$



$\Leftrightarrow$



Case II:  $\mathcal{S}_b$  nonempty. Let  $\vec{p} \in \mathcal{S}_b$  be any solution (of the inhomogeneous system).

Then  $\mathcal{S}_b = \mathcal{S}_0 + \vec{p} = \{ \vec{w} + \vec{p} \mid \vec{w} \in \mathcal{S}_0 \}$  is the translate of the homogeneous solution set by  $\vec{p}$ .

Why?

• If  $\vec{w} \in \mathcal{S}_0$ , then  $A(\vec{w} + \vec{p}) = A\vec{w} + A\vec{p} = \vec{0} + \vec{b} = \vec{b}$ .

So  $\mathcal{S}_0 + \vec{p} \subseteq \mathcal{S}_b$ .

• If  $\vec{x} \in \mathcal{S}_b$ , then write  $\vec{x} = (\vec{x} - \vec{p}) + \vec{p}$ . Since  $A(\vec{x} - \vec{p}) = A\vec{x} - A\vec{p} = \vec{b} - \vec{b} = \vec{0}$ , we see that

$\vec{x} - \vec{p} \in \mathcal{S}_0$ . So  $\vec{x} \in \mathcal{S}_0 + \vec{p}$ . This shows

the reverse inclusion  $\mathcal{S}_b \subseteq \mathcal{S}_0 + \vec{p}$ .

————— ◦ —————  
We now turn to the connection w/ linear independence.

Two equivalent conditions on columns  $\vec{v}_1, \dots, \vec{v}_n$  of an  $m \times n$  matrix  $A$ :

(I) The columns of  $A$  are linearly independent.

(II) All columns of  $\text{ref}(A)$  contain a leading '1'.

(i.e., all columns of  $A$  are pivot columns)

Check: (II)  $\Rightarrow$  (I): Suppose  $x_1 \vec{v}_1 + \dots + x_n \vec{v}_n = \vec{0}$ .

Since (II) holds, row-reducing  $\left[ A \mid \vec{0} \right]$

yields  $\left[ \begin{array}{cccc|c} 1 & & & & 0 \\ & \dots & & & 0 \\ & & \dots & & 0 \\ & & & 1 & 0 \\ & & & & \vdots \\ & & & & 0 \end{array} \right] \Rightarrow$  only solution is  $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ \vdots \\ x_n = 0 \end{cases}$

(I)  $\Rightarrow$  (II): If  $\text{ref} \left[ A \mid \vec{0} \right] = \left[ \begin{array}{cccc|c} 1 & \dots & & & 0 \\ & & \dots & & 0 \\ & & & \dots & 0 \\ & & & & \vdots \\ & & & & 0 \end{array} \right]$

has a non-pivot column (say, the  $i^{\text{th}}$ ),

then there exists a solution to  $\underbrace{x_1 \vec{v}_1 + \dots + x_n \vec{v}_n}_{\text{i.e. } A\vec{x}} = \vec{0}$

in which  $x_i$  can be anything we want (in particular, nonzero). This gives a linear dependence relation.



Ex 6 / Are the vectors  $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 6 \end{pmatrix}$   
linearly independent in  $\mathbb{R}^4$ ?

Row-reduce:

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} \boxed{1} & 0 & -\frac{1}{3} & \frac{4}{3} \\ 0 & \boxed{1} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \text{NO!}$$

[ To find a linear dependency, take for example  $x_3 = 3, x_4 = 0$ ,  
so  $x_1 = 1$  and  $x_2 = -2$ :  $1 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$  ] //

Notice that (as above,  $A = m \times n$  matrix):

(1) If  $m < n$  (more than  $m$  vectors in  $\mathbb{R}^m$ ), then  
The columns of  $A$  are never independent.

(Because there are only  $m$  rows to have leading '1's,  
and there are more columns than that.)

(2) If  $m = n$  (square matrix), then

columns are independent  $\iff \text{rref}(A) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

$\iff$  columns span  $\mathbb{R}^m$

(The last equivalence follows from Lecture 3, since the  $n \times n$  identity matrix  $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$  has no rows of "all 0" at the bottom.)

(3) If a set of vectors  $\vec{v}_1, \dots, \vec{v}_n$  contains  $\vec{0}$ , then that set is dependent.

(Say  $\vec{v}_1 = \vec{0}$ . Then  $1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \dots + 0 \cdot \vec{v}_n = \vec{0}$  is a linear dependency.)

(4) A set is linearly dependent  $\iff$  at least one of the  $\vec{v}_i$  is a linear combination of the others.

(If  $c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{0}$ , with some  $c_k \neq 0$ , then by rescaling we can assume  $c_k = 1$ . Then  $\vec{v}_k = -c_1 \vec{v}_1 - \dots - c_{k-1} \vec{v}_{k-1} - c_{k+1} \vec{v}_{k+1} - \dots - c_n \vec{v}_n$ .)

The converse is also clear from this.)

WARNING: Consider  $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ .  
 $\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4$

This is linearly dependent, and  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$  are each linear combinations of the others. But  $\vec{v}_4$  isn't. So

(4) means exactly what it says ("at least one", not "every")