

Lecture 5: Algebra with Matrices

Adding Matrices

- must be of same "dimensions" (both $m \times n$)
- add entry by entry

$$\text{Ex/ } \begin{pmatrix} 0 & 1 & -1 \\ 3 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 5 \end{pmatrix} //$$

- commutative : $A + B = B + A$
- associative : $(A + B) + C = A + (B + C)$
- additive identity : $A + 0 = A$, where $0 = \text{zero matrix}$
- additive inverse : $A + (-A) = 0$
- subtraction : $A - B := A + (-B)$
- cancellation : $A + B = C + B \Rightarrow A = C$

Scalar multiplication

- multiply each matrix entry by the scalar

$$\text{Ex/ } 3 \begin{pmatrix} 0 & 1 & -1 \\ 3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -3 \\ 9 & 0 & 6 \end{pmatrix} //$$

- distributivity : $c(A + B) = cA + cB$
 $(c+d)A = cA + dA$
- $(cd)A = c(dA)$

UPSHOT: $m \times n$ matrices form a vector space $\underline{M_{m,n}}$.

Multiplying matrices

- $A_{m \times n}$, $B_{k \times l}$: AB is defined when $k=n$
(and is then an $m \times l$ matrix)

- if $B = \begin{pmatrix} \overset{\uparrow}{\vec{w}_1} & \dots & \overset{\uparrow}{\vec{w}_k} \\ \downarrow & \dots & \downarrow \\ \vec{v}_1 & \dots & \vec{v}_n \end{pmatrix}$, then $AB := \begin{pmatrix} \overset{\uparrow}{\vec{w}_1} & \dots & \overset{\uparrow}{\vec{w}_k} \\ \downarrow & \dots & \downarrow \\ A\vec{w}_1 & \dots & A\vec{w}_k \end{pmatrix}$.

- So the q^{th} column of AB is

$$A\vec{w}_q = \begin{pmatrix} \overset{\uparrow}{\vec{v}_1} & \dots & \overset{\uparrow}{\vec{v}_n} \\ \downarrow & \dots & \downarrow \\ \vec{v}_1 & \dots & \vec{v}_n \end{pmatrix} \begin{pmatrix} b_{1q} \\ \vdots \\ b_{nq} \end{pmatrix} = b_{1q}\vec{v}_1 + \dots + b_{nq}\vec{v}_n \quad \left[\begin{array}{l} \text{column } q - \\ \text{vector interpretation} \end{array} \right]$$

$$= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{1q} \\ \vdots \\ b_{nq} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1j} b_{jq} \\ \vdots \\ \sum_{j=1}^n a_{mj} b_{jq} \end{pmatrix},$$

from which we see that the

$(p,q)^{\text{th}}$ entry of AB is

$$\underline{[AB]_{pq}} = \sum_{j=1}^n \underline{a_{pj}} \underline{b_{jq}}$$

- powers of a matrix: A^m means $\underbrace{A \cdot A \cdot \dots \cdot A}_{m \text{ times}}$

- identity matrix:

$$\mathbb{I}_n := \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} \overset{\uparrow}{\vec{e}_1} & \dots & \overset{\uparrow}{\vec{e}_n} \\ \downarrow & \dots & \downarrow \\ \vec{e}_1 & \dots & \vec{e}_n \end{pmatrix}$$

$$A \mathbb{I}_n = \begin{pmatrix} \uparrow & \uparrow \\ A\vec{e}_1 & \dots & A\vec{e}_n \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \dots & \vec{v}_n \\ \downarrow & \downarrow \end{pmatrix} = A = \mathbb{I}_m A$$

- distributivity : $A(\beta B + \gamma C) = \beta(AB) + \gamma(AC)$
 $(\beta B + \gamma C)A = \beta(BA) + \gamma(CA)$

- NOT commutative : $AB \neq BA$ in general

Ex/ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ //

- Does an $m \times n$ matrix A have a multiplicative inverse?

It depends on the matrix !

"right inverse"

- If $m > n$, there can't be an $n \times m$ B with $AB = \mathbb{I}_n$.

[Otherwise, for any $\vec{b} \in \mathbb{R}^m$ we'd have $A(B\vec{b}) = \mathbb{I}_m\vec{b} = \vec{b}$, which contradicts the fact that $\text{ref}(A)$ has rows of 0's at the bottom.] On the other hand, there could

be lots of matrices B with $BA = \mathbb{I}_n$.

"left inverse"

- For square matrices, as we'll see, a matrix with a left inverse also has a right inverse, & they are equal (and unique)!

We'll give a simple criterion below (in terms of RREF) for when a square matrix is invertible in this sense.

- If there aren't inverses, you can't "cancel":

Ex/ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$B \cdot A = \mathbf{0} = C \cdot A$$

but clearly $B \neq C$.

//

- Associativity holds : $(AB)C = A(BC)$.

If A $m \times n$, B $n \times r$, C $r \times s$, then

$$\begin{aligned}
 [(AB)C]_{i\lambda} &= \sum_k [AB]_{ik} C_{k\lambda} = \sum_k \left(\sum_j A_{ij} B_{jk} \right) C_{k\lambda} \\
 &= \sum_{j,k} (A_{ij} B_{jk}) C_{k\lambda} = \sum_{j,k} A_{ij} (B_{jk} C_{k\lambda}) \\
 &\quad \text{↑ } j,k \text{ since multiplication of real numbers is associative} \\
 &= \sum_j A_{ij} \left(\sum_k B_{jk} C_{k\lambda} \right) = \sum_j A_{ij} [BC]_{j\lambda} \\
 &= [A(BC)]_{i\lambda}.
 \end{aligned}$$

- Transpose of a matrix: A $m \times n \rightsquigarrow A^T$ $n \times m$, given by $[A^T]_{ij} := [A]_{ji}$. ($A = A^T \Leftrightarrow A$ is symmetric.)

Ex/ $\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}$ //

- $(AB)^T = B^T A^T$: $[(AB)^T]_{ij} = [AB]_{ji} = \sum_k a_{jk} b_{ki}$
- $$\begin{aligned}
 &= \sum_k b_{ki} a_{jk} = \sum_k [B^T]_{ik} [A^T]_{kj} \\
 &= [B^T A^T]_{ij}.
 \end{aligned}$$

Matrix Inverses

As we pointed out above, existence of multiplicative inverses of matrices is a messy issue for non-square matrices, and even square matrices may or may not have one. Let's begin our study of inverses in earnest with 2×2 matrices.

Ex / $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & ab-ab \\ cd & ad-bc \end{pmatrix} = (ad-bc) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So if $ad-bc \neq 0$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}_2,$$

making $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ "an inverse" to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on the right. //

- Some Questions : (1) Does it work on the left too?
(2) Is it unique?
(3) Is there an algorithm for producing it (not just for 2×2) ?

(As we'll see, the answer is YES for all three.)

Inversion Algorithm

$$A = \begin{pmatrix} \uparrow & & \uparrow \\ \downarrow & \dots & \downarrow \\ \vec{v}_1 & \dots & \vec{v}_n \end{pmatrix} \quad n \times n \text{ matrix}$$

Want an $n \times n$ $B = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{w}_1 & \dots & \vec{w}_n \\ \downarrow & & \downarrow \end{pmatrix}$ with $AB = I_n$, i.e.

$$\begin{pmatrix} \uparrow & & \uparrow \\ A\vec{w}_1 & \dots & A\vec{w}_n \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{e}_1 & \dots & \vec{e}_n \\ \downarrow & & \downarrow \end{pmatrix},$$

which is equivalent to solving n systems

$$(*) \quad A\vec{w}_1 = \vec{e}_1, \dots, A\vec{w}_n = \vec{e}_n$$

for $\vec{w}_1, \dots, \vec{w}_n$. Since $A\vec{w}_k = A \begin{pmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{pmatrix} = b_{1k}\vec{v}_1 + \dots + b_{nk}\vec{v}_n$, we must have that

$\text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$ contains $\vec{e}_1, \dots, \vec{e}_n$

$$\Rightarrow \text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \mathbb{R}^n$$

$\Rightarrow \text{ref}(A)$ has a leading 1 in each row

$$\Rightarrow \text{ref}(A) = I_n.$$

Conversely, if $\text{ref}(A) = I_n$, then (for each k)

$$\text{ref}[A | \vec{e}_k] = [I_n | \vec{c}_k]$$

and then $\begin{cases} x_1 = c_1 \\ \vdots \\ x_n = c_n \end{cases}$ solves the system $A\vec{x} = \vec{e}_k$ — that is, $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is the desired \vec{w}_k . To solve all n systems

simultaneously, take

$$\text{rref } [A \mid I_n] = [I_n \mid ?]$$

and then the "?" will give you B.

Ex / Given $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, we take rref of

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array} \right].$$

So $B = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ satisfies $AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$. //

Ex / $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ again. Write $\Delta = ad - bc$, assume $\Delta \neq 0$.

The row-reduction breaks into 2 cases : $a=0$ & $a \neq 0$. $I' \parallel$ do the $a \neq 0$ case :

$$\begin{aligned} [A \mid I_2] &\rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - \frac{cb}{a} & -\frac{c}{a} & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -\frac{c}{a} & \frac{d}{a} \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{bc}{a\Delta} + \frac{1}{a} & \frac{-b}{\Delta} \\ 0 & 1 & -\frac{c}{\Delta} & \frac{a}{\Delta} \end{array} \right], \text{ and } \frac{bc}{a\Delta} + \frac{1}{a} = \frac{bc + \Delta}{a\Delta} \\ &\Rightarrow B = \begin{pmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. // \end{aligned}$$

RREF Revisited

Recall the three types of row operations on a matrix:

- Replace
- Swap
- Scale

} claim that these can be achieved by left-multiplication by certain special matrices called elementary matrices.

Ex /

REPLACE

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

SWAP

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

SCALE

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\times \frac{1}{a}} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix}$$



Elementary Matrices

$$\begin{array}{c}
 \left(\begin{array}{cccccc} 1 & & & & & \\ & \ddots & & & & \\ & & a & & & \\ & & & \ddots & & 1 \\ & & & & \text{row } j \\ & & & & \text{column } i \end{array} \right) \cdot \left(\begin{array}{c} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_j \rightarrow \\ \vdots \\ \leftarrow \vec{r}_n \rightarrow \end{array} \right) = \left(\begin{array}{c} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_j + a\vec{r}_i \rightarrow \\ \vdots \\ \leftarrow \vec{r}_n \rightarrow \end{array} \right)
 \end{array}$$

$$\begin{array}{c}
 \left(\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \\ \text{row } i \\ \text{row } j \end{array} \right) \cdot \left(\begin{array}{c} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_i \rightarrow \\ \vdots \\ \leftarrow \vec{r}_j \rightarrow \\ \vdots \\ \leftarrow \vec{r}_n \rightarrow \end{array} \right) = \left(\begin{array}{c} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_i \rightarrow \\ \vdots \\ \leftarrow \vec{r}_j \rightarrow \\ \vdots \\ \leftarrow \vec{r}_n \rightarrow \end{array} \right)
 \end{array}$$

$$\begin{array}{c}
 \left(\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ & & \alpha & & \\ & & & 1 & \\ & & & & 1 \\ \text{row } i \end{array} \right) \cdot \left(\begin{array}{c} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_i \rightarrow \\ \vdots \\ \leftarrow \vec{r}_n \rightarrow \end{array} \right) = \left(\begin{array}{c} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \alpha \vec{r}_i \rightarrow \\ \vdots \\ \leftarrow \vec{r}_n \rightarrow \end{array} \right)
 \end{array}$$

The elementary matrices are the 3 kinds of matrices on the left, producing Replace, Swap, & Scale operations respectively. They are $n \times n$.

Upshot: The result of any sequence of row operations on any matrix A can be expressed as

$$E_N \cdot \dots \cdot E_1 \cdot A$$

where the E_i are elementary matrices.

Hence, if $\text{rref}(A) = I_n$, then

$$E \cdot A = I_n \quad (\text{where } E = \text{product of elementary matrices})$$

and

$\text{rref}[A | I_n] = E \cdot [A | I_n] = [EA | EI_n] = [I_n | E]$, giving another perspective on why the inversion algorithm works.

Conversely, if $(A = I_n \text{ (for some } n \times n \text{ matrix } C))$, then

$$A\vec{x} = \vec{0} \implies \vec{x} = CA\vec{x} = C\vec{0} = \vec{0}$$

(i.e. $A\vec{x} = \vec{0}$ has only the trivial solution) and A has no non-pivot columns. Hence $\text{rref}(A) = I_n$.

Moreover, if $CA = I_n = AB$, then

$$B = I_n B = CAB = CI_n = C.$$

This brings us to ...

Theorem/Definition: An $n \times n$ matrix A is invertible if (and only if) one of the equivalent statements

- (i) $\text{rref}(A) = I_n$
- (ii) there's a matrix B with $AB = I_n$
- (iii) " " " " C " $CA = I_n$

holds. The inverse of A is then $A^{-1} := B = C$.

Comments:

① Since elementary row operations transform A to the identity and are reversible, they also transform I_n to A . More precisely,

$$A = E_1^{-1} E_2^{-1} \cdots E_N^{-1},$$

so any invertible matrix is a product of elementary matrices.

② $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible $\Leftrightarrow ad - bc \neq 0$.

(We already knew " \Leftarrow " by an Example. If $ad - bc = 0$, then $ad = bc \Rightarrow$ rows are proportional $\Rightarrow \text{rref}(A) = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \Rightarrow A$ not invertible.)

Properties of the Inverse

Let $A, B = \text{invertible } n \times n \text{ matrices. Then}$

$$\bullet A^{-1} \cdot A = I_n = A \cdot A^{-1}$$

$$\bullet (A^{-1})^{-1} = A$$

$$\bullet (A^T)^{-1} = (A^{-1})^T \quad (\text{b/c } A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n)$$

$$\bullet (AB)^{-1} = B^{-1}A^{-1} \quad (\text{b/c } (AB) \cdot (B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I_n)$$

Using the inverse to solve inhomogeneous systems

Ex / Solve $\begin{pmatrix} 5 & -9 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Multiply both sides of $A\vec{x} = \vec{b}$ on the left by A^{-1} :

$$\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$= \underbrace{\frac{1}{5 \cdot 7 - (-4)(-9)} \begin{pmatrix} 7 & 9 \\ 4 & 5 \end{pmatrix}}_{A^{-1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= - \begin{pmatrix} 7 & 9 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = - \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad //$$

... goes a bit faster than row-reducing the augmented matrix!