II.B. Basis and dimension

How would you explain that a plane has two dimensions? Well, you can go in two independent directions, and no more. To make this idea precise, we formulate the

II.B.1. DEFINITION. A (finite) collection $\{\vec{v}_1, \ldots, \vec{v}_m\}$ in a vector space V (over a field F) is a **basis** of V if (a) they are linearly independent (over F), and (b) they span V (over F).[3]

The "over F" means that the linear combinations are understood to be of the form $\sum \alpha_i \vec{v}_i$, with $\alpha_i \in F$. As usual, our "default" will be $F = \mathbb{R}$.

II.B.2. EXAMPLE. A basis of $\mathbb{R}^m$ is given by the standard basis vectors $\{\hat{e}_i\}_{i=1}^m$.

Of course, one could say the same for any field: the standard basis vectors are also a basis for $\mathbb{C}^m$, viewed as a vector space over $\mathbb{C}$. But what if we view $\mathbb{C}^m$ as a vector space over $\mathbb{R}$? Then you need the $2m$ vectors $\hat{e}_1, \ldots, \hat{e}_m$ *and* $\mathbf{i}\hat{e}_1, \ldots, \mathbf{i}\hat{e}_m$! (Why?)

II.B.3. EXAMPLE. Let $P_2(x, y)$ denote the vector space of quadratic polynomials in two variables x and y, with real coefficients. This has the "monomial basis" $\{\rho_i^2(x, y)\}_{i=1}^6 = \{1,\ x,\ y,\ xy,\ x^2,\ y^2\}$. Similarly, $\{\rho_i^3(x, y)\}_{i=1}^{10} = \{1,\ x,\ y,\ xy,\ x^2,\ y^2,\ x^2y,\ y^2x,\ x^3,\ y^3\}$ gives a basis for the space $P_3(x, y)$ of cubic polynomials.[4]

We'll give a geometric application of this example below.

II.B.4. EXAMPLE. Let A be an $m \times n$ matrix, and V be the vector subspace of $\mathbb{R}^n$ consisting of solutions to $A\vec{x} = 0$. How do you compute a basis of V?

[3]This is not to say that you can't have infinite bases; indeed, non-finitely-generated vector spaces (like $P_\infty(x) :=$ all polynomials) may have such a basis. We're just not defining or discussing them at this stage.

[4]By "quadratic" resp. "cubic" polynomials, we mean polynomials of degree ≤ 2 resp. ≤ 3. (We include lower-degree polynomials to have closure under addition.)

By making use of the row-reduction algorithm, of course! For instance, we might have

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix} \longmapsto rref(A) = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The free variables are x_2 and x_4, and an arbitrary solution is given by plugging arbitrary values in for these and solving for the pivot variables x_1, x_3, x_5. To get a *basis*, you plug in $(x_2 = 1,\ x_4 = 0)$ and $(x_2 = 0,\ x_4 = 1)$ to obtain

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

More generally, the ith basis element is given by setting the ith free variable to 1 and the rest to zero.

II.B.5. EXAMPLE. Suppose A is an invertible $m \times m$ matrix. I claim that its columns $\{\vec{c}_i\}_{i=1}^m$ are a basis for $\mathbb{R}^m$.

To establish this, check that:

• $\underline{\{\vec{c}_i\} \text{ are linearly independent}}$: $\gamma_1\vec{c}_1 + \ldots + \gamma_m\vec{c}_m = 0 \implies A\vec{\gamma} = 0 \implies \vec{\gamma} = 0$ (since A is invertible).

• $\underline{\{\vec{c}_i\} \text{ span } \mathbb{R}^m}$: show every $\vec{y} \in \mathbb{R}^m$ can be written $\vec{y} = \gamma_1\vec{c}_1 + \ldots + \gamma_m\vec{c}_m\ (= A\vec{\gamma})$. This is easy: just put $\vec{\gamma} = A^{-1}\vec{y}$. □

We say that V is *finitely generated* if some finite subset $\{\vec{v}_1, \ldots, \vec{v}_m\}$ spans V. Here is the precise reason why dimension makes sense:

II.B.6. PROPOSITION. *If V is finitely generated then* (i) *it has a (finite) basis, and* (ii) *any* 2 *bases of V have the same number of elements.*

II.B.7. DEFINITION. We define the **dimension** $\dim_F(V)$ of V over F to be the number of elements in (ii).[5]

We get (i) by going through the generating subset $\{\vec{v}_1, \ldots, \vec{v}_m\}$ and throwing out any $\vec{v}_i$ that is a linear combination of the earlier vectors. What remains is a basis.[6]

In order to show (ii) we first prove the following

II.B.8. LEMMA. *If $\vec{v}_1, \ldots, \vec{v}_m$ span V then any linearly independent collection in V has no more than m vectors.*

PROOF. Consider any list $\vec{u}_1, \ldots, \vec{u}_n$, $n > m$ (of vectors in V). Since $\vec{v}_1, \ldots, \vec{v}_m$ span V, each $\vec{u}_j$ is a linear combination of the $\vec{v}_i$:

$$\vec{u}_j = \sum_{i=1}^{m} A_{ij}\vec{v}_i\,, \qquad \text{for } A \quad m \times n.$$

Since $m < n$, there is a nontrivial solution to $A\vec{x} = \vec{0}$, some *nonzero* $\vec{x} \in \mathbb{R}^n$. That is,

$$\sum_{j=1}^{n} A_{ij}x_j = 0 \qquad \text{for each } i,$$

and so

$$\sum_{j=1}^{n} x_j\vec{u}_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\vec{v}_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_j A_{ij}\right)\vec{v}_i = \sum_{i=1}^{m} 0\vec{v}_i = \vec{0}.$$

We conclude that $\{\vec{u}_j\}_{j=1}^{n}$ is *not* independent! □

Now let $\{\vec{v}_1, \ldots, \vec{v}_m\}$ and $\{\vec{w}_1, \ldots, \vec{w}_n\}$ be two bases for V. The $\{\vec{v}_i\}$ span V and the $\{\vec{w}_j\}$ are independent, so the Lemma implies that $n \leq m$. The same argument with $\{\vec{v}_i\}$ and $\{\vec{w}_j\}$ swapped yields $n \geq m$, and so we've proved Proposition II.B.6. We can rephrase the Lemma as follows: if $\dim(V) = m$ then (a) less than m vectors cannot span V, while (b) more than m vectors are linearly dependent.

[5]The F is typically omitted when there is no ambiguity. That said, in the first example we had $\dim_{\mathbb{C}}(\mathbb{C}^n) = n$ while $\dim_{\mathbb{R}}(\mathbb{C}^n) = 2n$.

[6]Any linear dependence relation $\sum_i a_i\vec{u}_i = 0$ on a list of vectors has a largest j for which $a_j \neq 0$. Dividing the relation by a_j then presents $\vec{u}_j$ as a linear combination of $\vec{u}_1, \ldots, \vec{u}_{n-1}$. So if no vector in a list is a linear combination of the vectors preceding it, the vectors are independent.

A Geometric Application. Given 5 points A, B, C, D, E "in general position" in the plane, there is a unique conic passing through them. Here's why: the equation of a conic Q has the form

$$0 = f_Q(x,y) = \sum_{i=1}^{6} a_i \rho_i^2(x,y) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2,$$

and the statement that it passes through A is

$$0 = f_Q(x_A, y_A) = \sum a_i \rho_i^2(x_A, y_A).$$

Continuing this for B thru E we get 5 equations in 6 unknowns a_i, a system which "generically"[7] has a "line" of solutions: if $(a_1, \ldots, a_6)$ solves the system then so does $(c \cdot a_1, \ldots, c \cdot a_6)$. But this just gives multiples of the same f_Q, which won't change the shape of the curve, and so there is really only one nontrivial solution: *through five general points in the plane there exists a unique conic Q.*

Similarly, given 8 points A, B, C, D, E, P_1, P_2, P_3 what can we say about the "vector space" of cubics passing through them (i.e. of cubic polynomials in (x,y) vanishing at all 8 points)? First of all, the vector space of all cubics

$$\sum_{i=1}^{10} a_i \rho_i^3(x,y) = a_1 + a_2 x + a_3 y + \ldots + a_9 x^3 + a_{10} y^3$$

should be thought of in terms of the coefficient vector $\vec{a}$. The 8 constraints give a linear system

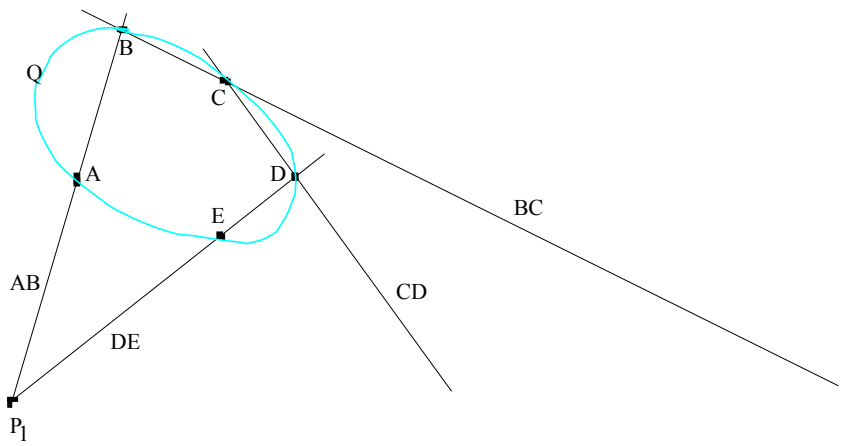
$$\left.\begin{array}{c} 0 = \sum_{i=1}^{10} a_i \rho_i^3(x_A, y_A) \\ \vdots \\ 0 = \sum_{i=1}^{10} a_i \rho_i^3(x_{P_3}, y_{P_3}) \end{array}\right\} 8 \text{ equations}$$

whose space of solutions "generically" is two-dimensional — there are 8 leading 1s in *rref* of the 8×10 $[\rho]$ matrix,[8] and therefore 2 parameters to choose freely.

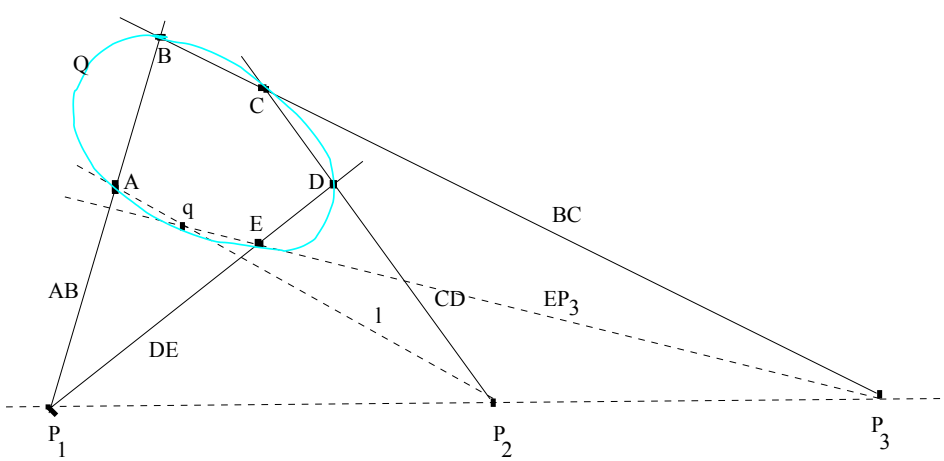
[7]More precisely, "generically" means here that the matrix $[\rho]$ (with rows indexed by A thru E, columns by $i = 1$ thru 6) is of maximal rank ($= 5$). One can show that this happens when no four of the five points are collinear.

[8]Once again, "generically" means that this matrix has maximal rank.

So start with A, B, C, D, E in "general position"; we'd like to construct points on the (unique) conic through them, using a straightedge. Begin by drawing the lines AB, DE, CD, and BC; label $AB \cap DE$ by P_1. (Q is drawn in a lighter color because we don't know what it looks like yet.)



Now draw (almost) any line ℓ through A — the choice is free except that ℓ shouldn't go through B, C, D, or E. All the rest of the construction depends on this choice; in particular the final point q depends on ℓ, and so by varying ℓ we can vary q and in this way (we hope) construct enough points on Q to get an idea of what it looks like. Label $\ell \cap CD =: P_2$; then draw P_1P_2 and label $P_1P_2 \cap BC =: P_3$; finally set $q := EP_3 \cap \ell$. We must prove $q \in Q$. Adding dotted lines to our diagram for the lines depending on ℓ,



Now consider the following three cubic equations:

$$\begin{aligned} f_1(x,y) &:= f_Q(x,y) \cdot f_{P_1P_2}(x,y) = 0, \\ f_2(x,y) &:= f_{AB}(x,y) \cdot f_{CD}(x,y) \cdot f_{EP_3}(x,y) = 0, \text{ and} \\ f_3(x,y) &:= f_\ell(x,y) \cdot f_{BC}(x,y) \cdot f_{DE}(x,y) = 0 \end{aligned}$$

(where $f_{AB}(x,y) = 0$ is the equation of the line AB, and so on). The graphical solutions of these equations are (respectively)

$$Q \cup P_1P_2\,, \quad AB \cup CD \cup EP_3\,, \quad \ell \cup BC \cup DE.$$

All 3 of these unions contain the 8 points A, B, C, D, E, P_1, P_2, P_3; that is, the cubic polynomials f_1, f_2, and f_3 each vanish at all 8 points. Above we explained that the (vector) space of such cubic polynomials was $10 - 8 = 2$ dimensional;[9] it follows that 3 "vectors" f_1, f_2, f_3 cannot be linearly independent and there is a nontrivial relation $\alpha f_1(x,y) + \beta f_2(x,y) + \gamma f_3(x,y) = 0$. Now we extract the information we want ($q \in Q$) from this relation.

First of all α cannot be zero. Otherwise $\beta f_2 = -\gamma f_3$ are multiples of each other, which means that $AB \cup CD \cup EP_3 = \ell \cup BC \cup DE$. That would imply ℓ was either AB, CD, or EP_3; and we said ℓ cannot pass through B, C, D, or E so this gives a contradiction.

So we may divide by $\alpha \implies f_1(x,y) = -\frac{\beta}{\alpha} f_2(x,y) - \frac{\gamma}{\alpha} f_3(x,y)$. Now since q is contained in both EP_3 and ℓ, $f_2(x,y)$ and $f_3(x,y)$ both vanish at $q = (x_q, y_q)$. (This is because f_2 contains a factor of f_{EP} and f_3 contains a factor of f_ℓ.) According to the relation, therefore also $f_1 = 0$ at q; this means that either f_Q or $f_{P_1P_2}$ vanishes at q, and so $q \in Q$ <u>*or*</u> P_1P_2. Suppose the latter: if $q \in P_1P_2$ then the three lines P_1P_2, EP_3, ℓ are the same, and so $E \in \ell$. But once again, we disallowed such a choice of ℓ at the outset. So $q \in Q$, and we're done.

Another way of thinking of this result is that *intercepts of opposite edges of a hexagon inscribed in a conic lie on line.* This is a theorem proved by Pascal in 1639, and known as his "mystic hexagon"!

[9] It is a (highly nontrivial) fact that the 8 points are "generic" if the first 5 are.

Exercises

(1) Let V be a vector space over a subfield F of the complex numbers. Suppose $\vec{u}, \vec{v}, \vec{w}$ are linearly independent vectors in V.
   (a) Prove that $\vec{u}+\vec{v}$, $\vec{v}+\vec{w}$, and $\vec{w}+\vec{u}$ are independent.
   (b) Explain why this implies that if $\vec{u}, \vec{v}, \vec{w}$ are a basis, then so are $\vec{u}+\vec{v}, \vec{v}+\vec{w}$, and $\vec{w}+\vec{u}$.
   (c) What happens if we take F to be $\mathbb{Z}/2\mathbb{Z}$ instead?

(2) Let V be the set of all 2×2 matrices A with *complex* entries which satisfy $A_{11} + A_{22} = 0$.
   (a) Show that V is a vector space over the field of *real* numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.
   (b) Find a basis for this vector space.
   (c) Let W be the set of all matrices A in V such that $A_{21} = -\overline{A_{12}}$ (the bar denotes complex conjugation). Prove that W is a subspace of V and find a basis for W.

(3) Prove that the space of all $m \times n$ matrices over the field F has dimension mn, by exhibiting a basis for this space.

(4) Let $P_3(x)$ denote the (real) vector space of polynomials (with real coefficients) of degree ≤ 3 in x. Do $2x^2+2x+1$, x^3+2x^2+2x+2, x^3-4x^2-3x-1, and x^3-2x^2-3x-4 constitute a basis of $P_3(x)$? [Hint: you may easily identify $P_3(x)$ with $\mathbb{R}^4$ (how?).]

(5) Given
$$A = \begin{pmatrix} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 2 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{pmatrix},$$
let V be the vector subspace of $\mathbb{R}^6$ consisting of solutions to $A\vec{x} = 0$. Find a basis of V.