V.C. Complex eigenvalues and other topics

Let's take a closer look at the characteristic polynomial for an $n \times n$ matrix. First consider $n = 2$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has

$$\begin{aligned} f_A(\lambda) &= \det(\lambda \mathbb{I} - A) \\ &= (\lambda - a)(\lambda - d) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - (\mathrm{tr} A)\lambda + (\det A). \end{aligned} \tag{V.C.1}$$

More generally, for $A \in M_n(\mathsf{F})$ one easily checks by direct calculation (see Exercise (1)) that

$$\begin{aligned} f_A(\lambda) &= \lambda^n - (\mathrm{tr} A)\lambda^{n-1} + P_2(A)\lambda^{n-2} - \ldots + (-1)^n \det A \\ &=: \sum_{k=0}^{n} (-1)^k P_k(A)\lambda^{n-k} \end{aligned} \tag{V.C.2}$$

where $P_1(A) = \mathrm{tr}(A)$, $P_n(A) = \det(A)$. The $P_k(A)$ so defined are are "invariant" polynomials (of degree k) in the entries of A:[7]

$$P_k(A) = P_k(S^{-1}AS), \tag{V.C.3}$$

as one can see by comparing coefficients of like powers of λ in the end terms of

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k P_k(A)\lambda^{n-k} &= \det(\lambda \mathbb{I} - A) = \det\{S^{-1}(\lambda \mathbb{I} - A)S\} \\ &= \det\{\lambda \mathbb{I} - S^{-1}AS\} = \sum_{k=0}^{n} (-1)^k P_k(S^{-1}AS)\lambda^{n-k}. \end{aligned}$$

If you didn't know $\mathrm{tr}(A) = \mathrm{tr}(S^{-1}AS)$ for all invertible S (say, by not doing Exercise III.D.3), now you do.

[7] This is short for "invariant under conjugation", where *conjugation* (by an invertible matrix M) sends $A \mapsto MAM^{-1}$. In other words, P_k does not distinguish between similar matrices (in the sense of Definition III.B.6).

In particular, if A is diagonalizable ($= SDS^{-1}$) then

$$P_k(A) = P_k(SDS^{-1}) = P_k(D). \tag{V.C.4}$$

Now by definition

$$f_D(\lambda) = \det \begin{pmatrix} \lambda - \lambda_1 & & \\ & \ddots & \\ & & \lambda - \lambda_n \end{pmatrix} = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$$
$$= \sum_{k=0}^{n} (-1)^k P_k(D) \lambda^{n-k},$$

and so

(V.C.5)

$$P_k(A) = \left\{ \begin{array}{c} \text{coefficient of } (-1)^k \lambda^{n-k} \\ \text{in } (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \end{array} \right\} = \sum_{1 \le i_1 < \ldots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$$
$$=: \text{“}k^{\text{th}} \text{ elementary symmetric function” in the } \{\lambda_i\}.$$

We caution that these are not formulas for the $P_k(A)$ any more than the special cases

(V.C.6)

$$\det(A) = P_n(A) = \lambda_1 \cdots \lambda_n\,, \qquad \operatorname{tr}(A) = P_1(A) = \lambda_1 + \ldots + \lambda_n$$

are formulas for determinant and trace. They also don't require A to be diagonalizable in order to hold — only that f_A can be written as a product $\prod_{i=1}^{n}(\lambda - \lambda_i)$ of linear factors, which is (for instance) always true when $\mathsf{F} = \mathbb{C}$.

Decoupling linear PDEs. Frequently in physics and engineering one encounters systems of partial differential equations like

$$\begin{aligned} \frac{\partial u_1}{\partial t} + a_{11}\frac{\partial u_1}{\partial x} + a_{12}\frac{\partial u_2}{\partial x} &= 0, \\ \frac{\partial u_2}{\partial t} + a_{21}\frac{\partial u_1}{\partial x} + a_{22}\frac{\partial u_2}{\partial x} &= 0 \end{aligned} \tag{V.C.7}$$

which are “coupled” in the sense that the equation for the change in u_1 has a term involving u_2, and vice versa. We can often decouple

them by a linear change in coordinates. Assume

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix}^{-1}$$
$$= SDS^{-1}$$

is diagonalizable, with eigenbasis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and rewrite the system in matrix form

$$\frac{\partial \vec{u}}{\partial t} + A\frac{\partial \vec{u}}{\partial x} = 0. \tag{V.C.8}$$

Let $\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = \vec{c} := S^{-1}\vec{u}$ be the coordinates of $\vec{u}$ with respect to $\mathcal{B}$, and write

$$\frac{\partial \vec{u}}{\partial t} + SDS^{-1}\frac{\partial \vec{u}}{\partial x} = 0 \quad \longrightarrow \quad S^{-1}\frac{\partial \vec{u}}{\partial t} + DS^{-1}\frac{\partial \vec{u}}{\partial t} = 0.$$

Since S^{-1} is a matrix of constants, the differential operators commute with it. The last equation becomes

$$\frac{\partial \vec{c}}{\partial t} + D\frac{\partial \vec{c}}{\partial x} = 0 \tag{V.C.9}$$

which corresponds to the two PDEs

$$\begin{aligned} \frac{\partial c_1}{\partial t} + \lambda_1 \frac{\partial c_1}{\partial x} &= 0, \\ \frac{\partial c_2}{\partial t} + \lambda_2 \frac{\partial c_2}{\partial x} &= 0. \end{aligned} \tag{V.C.10}$$

These are *not* coupled and may be solved separately to get the "traveling waves"

$$c_1(x,t) = \Phi_I(x - \lambda_1 t)\,, \quad c_2(x,t) = \Phi_{II}(x - \lambda_2 t) \tag{V.C.11}$$

where Φ_I, Φ_{II} are any functions on the real line. To get more specific we must specify the initial conditions

$$u_1(x,0) = g(x) \quad \text{and} \quad u_2(x,0) = h(x)$$

of the system.

V.C.12. EXAMPLE. Let's solve the system

$$\begin{aligned}\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial x}+4\frac{\partial u_2}{\partial x}&=0,\\ \frac{\partial u_2}{\partial t}+\frac{\partial u_1}{\partial x}+\ \ \frac{\partial u_2}{\partial x}&=0\end{aligned}$$

with initial conditions

$$u_1(x,0)=e^{-x}\,,\quad u_2(x,0)=\cos x.$$

The first step is to diagonalize

$$A=\begin{pmatrix}1&4\\1&1\end{pmatrix}=\begin{pmatrix}2&-2\\1&1\end{pmatrix}\cdot\begin{pmatrix}3&0\\0&-1\end{pmatrix}\cdot\frac{1}{4}\begin{pmatrix}1&2\\-1&2\end{pmatrix}=S\,D\,S^{-1}.$$

Then we solve for Φ_I and Φ_{II}:

$$\begin{pmatrix}\Phi_I(x)\\ \Phi_{II}(x)\end{pmatrix}=\begin{pmatrix}c_1(x,0)\\ c_2(x,0)\end{pmatrix}=S^{-1}\vec{u}(x,0)=\frac{1}{4}\begin{pmatrix}1&2\\-1&2\end{pmatrix}\begin{pmatrix}e^{-x}\\ \cos x\end{pmatrix}$$

i.e.

$$\Phi_I(x)=\frac{1}{4}e^{-x}+\frac{1}{2}\cos x\,,\quad \Phi_{II}(x)=-\frac{1}{4}e^{-x}+\frac{1}{2}\cos x\,;$$

and so (since $\lambda_1=3$, $\lambda_2=-1$)

$$\vec{c}(x,t)=\begin{pmatrix}\Phi_I(x-\lambda_1 t)\\ \Phi_{II}(x-\lambda_2 t)\end{pmatrix}=\begin{pmatrix}\frac{1}{4}e^{-(x-3t)}+\frac{1}{2}\cos(x-3t)\\ -\frac{1}{4}e^{-(x+t)}+\frac{1}{2}\cos(x+t)\end{pmatrix}.$$

This is the solution in eigen-coordinates. In terms of the original variables u_1 and u_2 we have

$$\begin{aligned}\begin{pmatrix}u_1\\u_2\end{pmatrix}&=\vec{u}(x,t)=S\vec{c}(x,t)\\ &=\begin{pmatrix}2&-2\\1&1\end{pmatrix}\begin{pmatrix}\frac{1}{4}e^{-(x-3t)}+\frac{1}{2}\cos(x-3t)\\ -\frac{1}{4}e^{-(x+t)}+\frac{1}{2}\cos(x+t)\end{pmatrix}\\ &=\begin{pmatrix}\frac{1}{2}e^{-(x-3t)}+\cos(x-3t)+\frac{1}{2}e^{-(x+t)}-\cos(x+t)\\ \frac{1}{4}e^{-(x-3t)}+\frac{1}{2}\cos(x-3t)-\frac{1}{4}e^{-(x+t)}+\frac{1}{2}\cos(x+t)\end{pmatrix}.\end{aligned}$$

If you are interested in applications of this sort of thing, look in any book on applied PDEs (a beautiful subject in itself).

Complex diagonalization of real matrices. In the rest of this section we consider a matrix $A \in M_n(\mathbb{R})$ with real entries. If the characteristic polynomial $f_A(\lambda)$ does not break completely into linear factors over $\mathbb{R}$, then A does not have a complete set of eigenvalues in $\mathbb{R}$. (The algebraic multiplicities k_i of the *real* eigenvalues do not add to n.) In this case $f_A(\lambda) =$

$$(\lambda - \sigma_1)^{k_1} \cdots (\lambda - \sigma_r)^{k_r} \cdot \underbrace{(\lambda^2 + \mu_1\lambda + \nu_1)^{\ell_1} \cdots (\lambda^2 + \mu_s\lambda + \nu_s)^{\ell_s}}_{\text{irreducible quadratic factors: } \mu_k^2 < 4\nu_k}$$

where σ_i, μ_j, ν_j are all real and $\sum k_j + 2\sum \ell_k = n$.

Over $\mathbb{C}$ this $f_A(\lambda)$ factors completely: if we set $\xi_j = -\frac{\mu_j}{2}$, $\eta_j = -\frac{1}{2}\sqrt{4\nu_j - \mu_j^2}$ (since $\mu_j^2 < 4\nu_j$ by hypothesis, these will be real) then the quadratic formula shows

$$(\lambda^2 + \mu_j\lambda + \nu_j) \;=\; \big(\lambda - (\xi_j + \mathbf{i}\eta_j)\big)\,\big(\lambda - (\xi_j - \mathbf{i}\eta_j)\big)\,.$$

Therefore each irreducible quadratic factor gives rise to a *conjugate pair* of complex eigenvalues, and the algebraic multiplicities of

$$\sigma_1\,,\,\ldots\,,\,\sigma_s\,;\,\xi_1 + \mathbf{i}\eta_1, \xi_1 - \mathbf{i}\eta_1, \ldots, \xi_t + \mathbf{i}\eta_t, \xi_t - \mathbf{i}\eta_t$$

do add to n.

Since A is real, if $A\vec{v} = \lambda\vec{v}$ for a real $\vec{v} \neq 0$ then λ is also real. So if we now take a (nonzero) eigenvector $\vec{v} \in E_{\xi_j + \mathbf{i}\eta_j}(A)$, this $\vec{v}$ *cannot* be real since $\eta_j \neq 0$ by hypothesis. Therefore we write $\vec{v} = \vec{u} + \mathbf{i}\vec{w}$, where $\vec{u}$ and $\vec{w}$ are real and $\vec{w} \neq 0$. We have

$$\{(\xi_j + \mathbf{i}\eta_j)\mathbb{I} - A\}(\vec{u} + \mathbf{i}\vec{w}) = 0\,;$$

and taking the complex conjugate yields

$$\{(\xi_j - \mathbf{i}\eta_j)\mathbb{I} - A\}(\vec{u} - \mathbf{i}\vec{w}) = 0,$$

where A is unaffected because it is real. That is, we have

$$\overline{\vec{v}} = \vec{u} - \mathbf{i}\vec{w} \;\in E_{\xi_j - \mathbf{i}\eta_j}(A),$$

and so to each conjugate pair of complex eigenvalues of A there is at least one conjugate pair of complex eigen*vectors*.

In fact, *if* A is diagonalizable over $\mathbb{C}$, then there is an eigenbasis

$$\vec{v}_1, \ldots, \vec{v}_s\,;\ \vec{u}_1 + \mathbf{i}\vec{w}_1, \vec{u}_1 - \mathbf{i}\vec{w}_1,\ \ldots,\ \vec{u}_t + \mathbf{i}\vec{w}_t, \vec{u}_t - \mathbf{i}\vec{w}_t$$

(where all $\vec{v}_i$, $\vec{u}_j$, $\vec{w}_j$ are real) with respective eigenvalues

$$\lambda_1, \ldots, \lambda_s\,;\ \xi_1 + \mathbf{i}\eta_1,\ \xi_1 - \mathbf{i}\eta_1\,,\ \ldots,\ \xi_t + \mathbf{i}\eta_t, \xi_t - \mathbf{i}\eta_t$$

(λ_i, ξ_j, $\eta_j \in \mathbb{R}$) where $s + 2t = n$. Of course one can now write a matrix S with the vectors of the (complex) eigenbasis as its columns, and get $A = SDS^{-1}$ where D has the (complex) eigenvalues as its diagonal entries. Such a diagonalization is old news. The new, beautiful fact is that one can decompose A into *real* $n \times n$ matrices $A = MBM^{-1}$ where

$$M = \begin{pmatrix} \uparrow & & \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow \\ | & & | & | & | & & | & | \\ | & & | & | & | & & | & | \\ \vec{v}_1 & \cdots & \vec{v}_s & \vec{u}_1 & -\vec{w}_1 & \cdots & \vec{u}_t & -\vec{w}_t \\ | & & | & | & | & & | & | \\ | & & | & | & | & & | & | \\ \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \end{pmatrix},$$

$$B = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_s & & & & & \\ & & & \xi_1 & -\eta_1 & & & \\ & & & \eta_1 & \xi_1 & & & \\ & & & & & \ddots & & \\ & & & & & & \xi_t & -\eta_t \\ & & & & & & \eta_t & \xi_t \end{pmatrix}$$

— i.e. B is "almost diagonal". You can think of the 2×2 blocks along the diagonal as corresponding to the irreducible quadratic factors of $f_A(\lambda)$. To see why this works, we turn to a more detailed discussion of the $n = 2$ case.

The situation for real 2×2 matrices. Let's start by diagonalizing (real) *rotation-dilation* matrices over $\mathbb{C}$. That is, we consider matrices of the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in M_2(\mathbb{R}).$$

Recall that these factor as rR_θ, where R_θ is the counterclockwise rotation by $\theta = \arctan(b/a)$ and $r = \sqrt{a^2+b^2}$ the (scalar) dilation. From Example V.B.11, the characteristic polynomial f_A has roots $a \pm b\mathbf{i} = re^{\pm \mathbf{i}\theta}$, a conjugate pair of *complex* eigenvalues.

To obtain the eigenvectors, look at

$$\begin{aligned} E_{a+b\mathbf{i}}(A) &= \ker\{(a+b\mathbf{i})\mathbb{I} - A\} = \ker \begin{pmatrix} b\mathbf{i} & b \\ -b & b\mathbf{i} \end{pmatrix} \\ &= \ker \begin{pmatrix} \mathbf{i} & 1 \end{pmatrix} = span \begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} E_{a-b\mathbf{i}}(A) &= \ker\{(a-b\mathbf{i})\mathbb{I} - A\} = \ker \begin{pmatrix} -b\mathbf{i} & b \\ b & -b\mathbf{i} \end{pmatrix} \\ &= \ker \begin{pmatrix} -\mathbf{i} & 1 \end{pmatrix} = span \begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix}. \end{aligned}$$

(Note that you could also just take the conjugate of $\left(\begin{smallmatrix} 1 \\ -\mathbf{i} \end{smallmatrix}\right)$ to get $\left(\begin{smallmatrix} 1 \\ \mathbf{i} \end{smallmatrix}\right)$.) Instead of S, write

$$N := \begin{pmatrix} 1 & 1 \\ -\mathbf{i} & \mathbf{i} \end{pmatrix}; \tag{V.C.13}$$

N will mean this particular matrix for the remainder of the section. By the usual diagonalization procedure one gets

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = NDN^{-1} = \begin{pmatrix} 1 & 1 \\ -\mathbf{i} & \mathbf{i} \end{pmatrix} \cdot \begin{pmatrix} a+b\mathbf{i} & 0 \\ 0 & a-b\mathbf{i} \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & \mathbf{i} \\ 1 & -\mathbf{i} \end{pmatrix}.$$

From this we have immediately the useful relation

$$\boxed{\begin{pmatrix} a+b\mathbf{i} & 0 \\ 0 & a-b\mathbf{i} \end{pmatrix} = N^{-1} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} N = rN^{-1}R_\theta N.} \tag{V.C.14}$$

More generally, let

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in M_2(\mathbb{R});$$

then there are three possibilities for the roots of

$$\begin{aligned} f_A(\lambda) &= \lambda^2 - (\alpha_{11} + \alpha_{22})\lambda + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) \\ &= \lambda^2 - (\mathrm{tr}A)\lambda + \det A, \end{aligned}$$

based on its discriminant (i.e., the quantity under the radical in the quadratic formula):

(i) $\underline{(\mathrm{tr}A)^2 > 4\det A:}$
distinct real roots $\implies$ A is diagonalizable over $\mathbb{R}$;

(ii) $\underline{(\mathrm{tr}A)^2 = 4\det A:}$
repeated real root $\implies$???
(A may not be diagonalizable even over $\mathbb{C}$);

(iii) $\underline{(\mathrm{tr}A)^2 < 4\det A:}$
(distinct) conjugate complex roots $\implies$
A is diagonalizable over $\mathbb{C}$ but *not* over $\mathbb{R}$
($f_A(\lambda)$ is an "irreducible" quadratic).

We are interested in case (iii). Call the roots $a + b\mathbf{i}$, $a - b\mathbf{i}$ (noting that $b \neq 0$), and let

$$\vec{v} = \vec{u} + \mathbf{i}\vec{w} \in E_{a+b\mathbf{i}}(A)$$

be a nonzero eigenvector (in particular, $\vec{w} \neq 0$ as before). Noting that A is real, so $\overline{A} = A$, we have

$$A(\vec{u} + i\vec{w}) = (a + b\mathbf{i})(\vec{u} + \mathbf{i}\vec{w}) \quad \overset{\text{conjugation}}{\longrightarrow} \quad A(\vec{u} - \mathbf{i}\vec{w}) = (a - b\mathbf{i})(\vec{u} - \mathbf{i}\vec{w}).$$

Consequently

$$\vec{u} - \mathbf{i}\vec{w} \in E_{a-\mathbf{i}b}(A)$$

is another eigenvector of A, and $\{\vec{u} + \mathbf{i}\vec{w}, \vec{u} - \mathbf{i}\vec{w}\}$ an A-eigenbasis. (They are independent because they have distinct eigenvalues; note also that their independence implies that of $\vec{u}$ and $\vec{w}$.)

Setting

$$S = \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} + \mathbf{i}\vec{w} & \vec{u} - \mathbf{i}\vec{w} \\ \downarrow & \downarrow \end{pmatrix},$$

we have by (V.C.14)

$$A = S \begin{pmatrix} a + \mathbf{i}b & 0 \\ 0 & a - \mathbf{i}b \end{pmatrix} S^{-1} = SN^{-1} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} NS^{-1}.$$

Thus, if we set

$$M = SN^{-1} = \frac{1}{2} \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} + \mathbf{i}\vec{w} & \vec{u} - \mathbf{i}\vec{w} \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} 1 & \mathbf{i} \\ 1 & -\mathbf{i} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \uparrow & \uparrow \\ \left\{ \begin{matrix} (\vec{u} + \mathbf{i}\vec{w}) \\ +(\vec{u} - \mathbf{i}\vec{w}) \end{matrix} \right\} & \left\{ \begin{matrix} \mathbf{i}(\vec{u} + \mathbf{i}\vec{w}) \\ -\mathbf{i}(\vec{u} - \mathbf{i}\vec{w}) \end{matrix} \right\} \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} & -\vec{w} \\ \downarrow & \downarrow \end{pmatrix},$$

then we obtain a decomposition
(V.C.15)

$$A = M \begin{pmatrix} a & -b \\ b & a \end{pmatrix} M^{-1} = r \cdot MR_\theta M^{-1}, \quad M = \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} & -\vec{w} \\ \downarrow & \downarrow \end{pmatrix}$$

entirely over $\mathbb{R}$!! This plays exactly the same role in solving linear dynamical systems $/\mathbb{R}$ when the eigenvalues are conjugate complex, as the decomposition $A = SDS^{-1}$ (diagonalization) did in the case of distinct real eigenvalues.

V.C.16. EXAMPLE. Take

$$A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix},$$

so that

$$f_A(\lambda) = \det \begin{pmatrix} \lambda - 3 & 5 \\ -1 & \lambda + 1 \end{pmatrix} = \lambda^2 - 2\lambda + 2$$

has conjugate complex roots $\lambda = 1 \pm \mathbf{i} =: a \pm \mathbf{i}b$, which is to say $a = b = 1$. Moreover

$$E_{1+\mathbf{i}}(A) = \ker \begin{pmatrix} -2+\mathbf{i} & 5 \\ -1 & 2+\mathbf{i} \end{pmatrix}$$

$$= \ker \begin{pmatrix} -1 & 2+\mathbf{i} \end{pmatrix} = span \begin{pmatrix} 2+\mathbf{i} \\ 1 \end{pmatrix},$$

and so we have the eigenvector

$$\vec{v} = \begin{pmatrix} 2+\mathbf{i} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mathbf{i} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: \vec{u} + \mathbf{i}\vec{w}\,;$$

its conjugate $\vec{u} - \mathbf{i}\vec{w}$ is the other one. Set

$$M = \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} & -\vec{w} \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

and note that

$$M^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix};$$

then according to our result above, we have the *real* decomposition

$$A = M \begin{pmatrix} a & -b \\ b & a \end{pmatrix} M^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

One may conceptualize this in terms of transformations as follows: if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation with matrix $[T]_{\hat{e}} = A$ relative to the standard basis $\{\hat{e}_1, \hat{e}_2\}$ of $\mathbb{R}^2$, then in terms of $\mathcal{B} = \{\vec{u}, -\vec{w}\}$

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sqrt{2}R_{\pi/4} =: rR_\theta.$$

So T is a sort of "skew" rotation-dilation.

Real dynamical systems with complex eigenvalues.

(I). *A discrete dynamical system.* Let A remain the matrix of the above example. We will work out the solution to

$$\vec{x}(\tau+1) = A\vec{x}(\tau)$$

abstractly and then substitute in the numerical information. let

$$\vec{\gamma}(\tau) = M^{-1}\vec{x}(\tau), \quad \vec{\gamma}_0 = M^{-1}\vec{x}_0$$

be the present analogue of the eigen-coordinates ($\vec{c} = S^{-1}\vec{x}$). These are the coordinates with respect to $\vec{u}$ and $-\vec{w}$: that is, $\vec{x} = \gamma_1\vec{u} - \gamma_2\vec{w}$. Using our real decomposition of A (because we want a real solution),

$$\begin{aligned} \vec{x}(\tau) &= A^\tau\vec{x}_0 = (rMR_\theta M^{-1})^\tau\vec{x}_0 = r^\tau M(R_\theta)^\tau M^{-1}\vec{x}_0 \\ &= r^\tau MR_{\tau\theta}M^{-1}\vec{x}_0. \end{aligned} \tag{V.C.17}$$

Multiplying both sides by M^{-1}, this becomes

$$M^{-1}\vec{x}(\tau) = r^\tau R_{\tau\theta}M^{-1}\vec{x}_0$$

or

$$\vec{\gamma}(\tau) = r^\tau R_{\tau\theta}\vec{\gamma}_0.$$

Suppose we start at $\vec{u}$, i.e take $\vec{x}_0 = \vec{u} = \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)$. Then $\vec{\gamma}_0 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ (why?) and

$$\vec{\gamma}(\tau) \;=\; (\sqrt{2})^\tau R_{(\frac{\pi}{4}\tau)}\begin{pmatrix}1\\0\end{pmatrix} \;=\; \sqrt{2}^\tau\begin{pmatrix}\cos\frac{\pi}{4}\tau\\ \sin\frac{\pi}{4}\tau\end{pmatrix},$$

or in $\hat{e}$-coordinates

$$\vec{x}(\tau) = M\vec{\gamma}(\tau) = \sqrt{2}^\tau\left\{\cos(\frac{\pi}{4}\tau)\vec{u} - \sin(\frac{\pi}{4}\tau)\vec{w}\right\}.$$

(II). *A continuous dynamical system.* Again assuming $A =$ the matrix from the example, let's investigate the *continuous* dynamical system

$$\frac{d\vec{x}}{dt} = A\vec{x}(t).$$

This turns out to be unexpectedly complicated; we will employ *both* sets of coordinates. Put

$$\vec{\gamma}(t) = M^{-1}\vec{x}(t), \qquad \vec{c}(t) = S^{-1}\vec{x}(t) \qquad \text{where}$$

$$M = \begin{pmatrix}\uparrow & \uparrow\\ \vec{u} & -\vec{w}\\ \downarrow & \downarrow\end{pmatrix}, \qquad S = \begin{pmatrix}\uparrow & \uparrow\\ \vec{u}+\mathbf{i}\vec{w} & \vec{u}-\mathbf{i}\vec{w}\\ \downarrow & \downarrow\end{pmatrix}.$$

First use the eigencoordinates $\vec{c} = \left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right)$ to transform

$$\frac{d\vec{x}}{dt} = S \begin{pmatrix} a + b\mathbf{i} & 0 \\ 0 & a - b\mathbf{i} \end{pmatrix} S^{-1}\vec{x}(t)$$

to

$$\frac{d\vec{c}}{dt} = \begin{pmatrix} a + b\mathbf{i} & 0 \\ 0 & a - b\mathbf{i} \end{pmatrix} \vec{c}(t).$$

The solution to this is just

$$\vec{c}(t) \;=\; \begin{pmatrix} e^{(a+b\mathbf{i})\tau} & 0 \\ 0 & e^{(a-b\mathbf{i})t} \end{pmatrix} \vec{c}_0 \;=\; e^{at} \begin{pmatrix} e^{\mathbf{i}bt} & 0 \\ 0 & e^{-\mathbf{i}bt} \end{pmatrix} \vec{c}_0\,,$$

i.e.

$$S^{-1}\vec{x}(t) \;=\; e^{at} \begin{pmatrix} e^{\mathbf{i}bt} & 0 \\ 0 & e^{-\mathbf{i}bt} \end{pmatrix} S^{-1}\vec{x}_0$$

or

$$\begin{aligned} \vec{x}(t) \;&=\; e^{at} S \begin{pmatrix} e^{\mathbf{i}bt} & 0 \\ 0 & e^{-\mathbf{i}bt} \end{pmatrix} S^{-1}\vec{x}_0 \\ &=\; e^{at} S \begin{pmatrix} \cos bt + \mathbf{i}\sin bt & 0 \\ 0 & \cos bt - \mathbf{i}\sin bt \end{pmatrix} S^{-1}\vec{x}_0 \\ &=\; e^{at} S N^{-1} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} N S^{-1}\vec{x}_0 \\ &=\; e^{at} M R_{bt} M^{-1}\vec{x}_0. \end{aligned} \tag{V.C.18}$$

In order to decouple (and solve) the differential equations we had to diagonalize, but that introduced complex numbers. To then get rid of them, we had to be more clever than in the last three types of dynamical systems (discrete with real *or* complex λ, continuous with real λ).

Now all we have to do is transform to $\{\vec{u}, -\vec{w}\}$ coordinates:

$$M^{-1}\vec{x}(t) \;=\; e^{at} R_{bt} M^{-1}\vec{x}_0 \quad\longrightarrow\quad \vec{\gamma}(t) = e^{at} R_{bt}\vec{\gamma}_0.$$

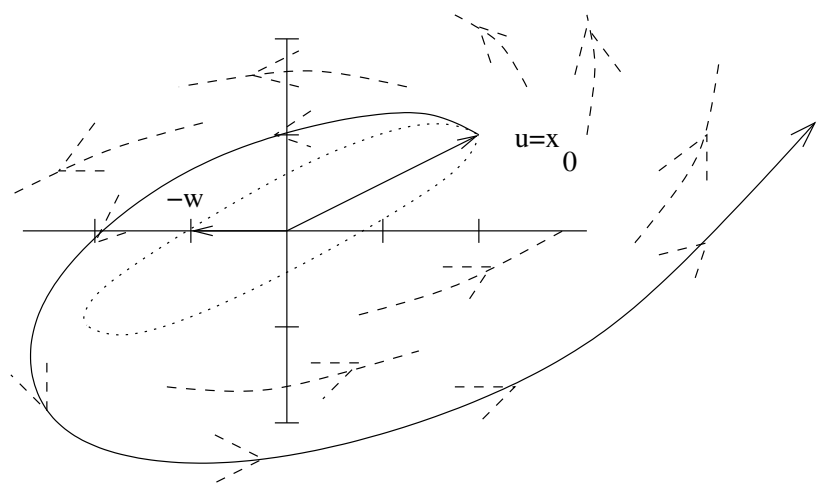
Again let $\vec{x}_0 = \binom{2}{1} \leftrightarrow \vec{\gamma}_0 = \binom{1}{0}$ be the initial state, so that the solution looks like

$$\vec{\gamma}(t) = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{at} \begin{pmatrix} \cos bt \\ \sin bt \end{pmatrix}$$

or

$$\begin{aligned} \vec{x}(t) = M\vec{\gamma}(t) &= e^{at} \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} & -\vec{w} \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} \cos bt \\ \sin bt \end{pmatrix} \\ &= e^{at} \left\{ \cos(bt)\vec{u} - \sin(bt)\vec{w} \right\} \\ &\boxed{= e^{t} \left\{ \cos(t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.} \end{aligned}$$

This corresponds to the picture



Although for this particular matrix there are not dramatic differences between the discrete and continuous systems as far as the *qualitative* phase diagram goes, note the distinctions in the dilations

$$r^{\tau} = \sqrt{a^2 + b^2}^{\tau} \qquad \text{vs.} \qquad e^{at}$$

and the rotations

$$R_{\tau \cdot \arctan(b/a)} \qquad \text{vs.} \qquad R_{bt}.$$

This leads to qualitative differences for certain values of a and b.

Exercises

(1) Check (V.C.2): (a) For $P_n = \det$, set $\lambda = 0$ to get the constant term of f_A; (b) for $P_1 = \mathrm{tr}$, argue that only the "diagonal" pattern contributes to the coefficient of λ^{n-1} in f_A.

(2) (a) Show

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

cannot be diagonalized over $\mathbb{R}$. (A 90° rotation in 3 space has only one real eigenvector — the one it rotates around.)

(b) Now diagonalize it over $\mathbb{C}$.

(3) Decouple the linear system

$$\frac{\partial u}{\partial t} + 8\frac{\partial v}{\partial x} = 0$$
$$\frac{\partial v}{\partial t} + 2\frac{\partial u}{\partial x} = 0$$

of PDEs and find the solution given initial conditions $u(x,0) = \sin x$, $v(x,0) = x^2$.

(4) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given in the standard basis by

$$A = \begin{pmatrix} -4 & 10 \\ -4 & 8 \end{pmatrix}.$$

(a) Diagonalize A over $\mathbb{C}$.

(b) Now show that A is similar ($A = R^{-1}BR$) to a rotation-dilation matrix (both R and B are supposed to be real).

(c) Give a closed-form solution to the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ if $\vec{x}(0) = \binom{3}{2}$ and draw a phase portrait for the system.

(5) "Partially diagonalize" the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 3 \end{pmatrix}$$

over $\mathbb{R}$: that is, write it as MBM^{-1} where

$$B = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix}$$

with $a, b, \lambda \in \mathbb{R}$.