VI. Canonical forms

VI.A. The minimal polynomial of a transformation

The statement that $\vec{v}_0$ is an eigenvector of $A \in M_n(\mathbb{R})$ with eigenvalue 3 can be written

$$(3\mathbb{I} - A)\vec{v}_0 = 0\,.$$

That is, if you plug A into the polynomial $3 - x$, then the resulting matrix annihilates $\vec{v}_0$. Is there a corresponding statement for *all* vectors $\vec{v} \in \mathbb{R}^n$? That is, a polynomial into which we may plug A to get the *zero* matrix (which is the only matrix annihilating all vectors)?

Consider $M_n(\mathbb{R})$ as a vector space over $\mathbb{R}$ of dimension n^2.[1] Apparently the $n^2 + 1$ "vectors"

$$\mathbb{I},\ A,\ A^2,\ \ldots,\ A^{(n^2)}$$

cannot all be independent. So there is a relation

$$\alpha_0\mathbb{I} + \alpha_1 A + \alpha_2 A^2 + \ldots + \alpha_{(n^2)} A^{(n^2)} = 0,$$

where not all α_i are zero. That is, A is "annihilated" by a polynomial $q(x)$ of degree n^2, in the sense that $q(A)$ is the zero matrix.

However we should (at least some of the time) be able to do better than this. If A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$(\lambda_1\mathbb{I} - A)(\lambda_2\mathbb{I} - A)\cdots(\lambda_n\mathbb{I} - A)\vec{v} = 0$$

for all $\vec{v} \in \mathbb{R}^n$. (Write $\vec{v} = \beta_1\vec{v}_1 + \ldots + \beta_n\vec{v}_n$ in terms of the eigenbasis; then use the fact that all the $(\lambda_i\mathbb{I} - A)$ commute with one another.) Multiplying this out gives a polynomial in A of degree n, not n^2.

[1] The "standard" basis of this vector space would be the matrices with 1 in the ij^{th} place and 0's in the other places, $i, j = 1, \ldots, n$. Clearly there are n^2 of these.

Similarly, if A is of the (non-diagonalizable!) form

$$\begin{pmatrix} 0 & 1 & & * \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

then we find it satisfies $A^n = 0$ (A is *nilpotent*). In fact, the characteristic polynomials ($= \det(\lambda\mathbb{I} - A)$) in these two cases are

$$(\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \qquad \text{and} \qquad \lambda^n,$$

so the following more general result should not surprise you:

VI.A.1. THEOREM (Cayley-Hamilton). *Let $A \in M_n(\mathsf{F})$ be a square matrix over any field. Then A is annihilated by its own characteristic polynomial, i.e. if $f_A(\lambda) := \det(\lambda\mathbb{I} - A)$ then $f_A(A) = 0$.*

VI.A.2. REMARK. Thus we can always do much better than n^2, since $\deg f_A = n$. However, the proof is not as easy as

$$\det(A\mathbb{I} - A) = \det 0 = 0.$$

This is cheating. Substituting in A before you take the determinant is not the same as doing so *after* taking the determinant. We now give two correct proofs.

FIRST PROOF OF VI.A.1. We need to introduce (a little more consciously than before) matrices whose entries are *polynomials in λ*. Let $\mathsf{F}[\lambda]$ denote polynomials of arbitrary degree in λ with coefficients in F, and consider $M \in M_n(\mathsf{F}[\lambda])$. The tricky thing is that we must *avoid* dividing by λ — polynomials are not invertible like real numbers.

One definition that involved no inverting of anything was that of the adjugate of A, whose ij^{th} entry was defined to be

$$\det\{ji^{\text{th}} \text{ minor of } A\} \times (-1)^{i+j}.$$

In §IV.C, we had from Cramer's rule (assuming $A \in M_n(\mathsf{F})$, not $M_n(\mathsf{F}[\lambda])$) the relationship

$$A^{-1} = \frac{1}{\det A}\mathrm{adj}A\,, \quad \text{i.e. } A\left(\frac{1}{\det A}\mathrm{adj}A\right) = \mathbb{I}.$$

Clearing the denominator yields something which still holds[2] for M:

$$M(\mathrm{adj}M) = (\det M)\mathbb{I}.$$

Now let $M = \lambda\mathbb{I} - A$. We have

$$(\lambda\mathbb{I} - A)[\mathrm{adj}(\lambda\mathbb{I} - A)] = \det(\lambda\mathbb{I} - A)\mathbb{I} = f_A(\lambda) \cdot \mathbb{I}.$$

One may decompose any $M \in M_n(\mathsf{F}[\lambda])$ into powers of λ, $M = \sum \lambda^k B_k$ where $B_k \in M_n(\mathsf{F})$: for example,

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We do this for

$$\mathrm{adj}(\lambda\mathbb{I} - A) = \sum_{k=0}^{n-1} \lambda^k S_k,$$

and write also

$$f_A(\lambda) = \sum_{j=0}^{n} a_j \lambda^j.$$

We have

$$(\lambda\mathbb{I} - A)\left(\sum_{k=0}^{n-1} \lambda^k S_k\right) = \sum_{j=0}^{n} a_j \lambda^j \mathbb{I}$$

or

$$\begin{aligned} & -AS_0 + \lambda(S_0 - AS_1) + \lambda^2(S_1 - AS_2) + \ldots \\ & \qquad\qquad + \lambda^{n-1}(S_{n-2} - AS_{n-1}) + \lambda^n S_{n-1} \\ & = \; a_0\mathbb{I} + a_1\lambda\mathbb{I} + a_2\lambda^2\mathbb{I} + \ldots + a_{n-1}\lambda^{n-1}\mathbb{I} + a_n\lambda^n\mathbb{I}. \end{aligned}$$

[2]Technically, applying Cramer as we did in §IV.C requires knowing A is invertible, which typically won't be true for M. See Exercise (5) below for a way around this.

You may equate "coefficients" of like powers of λ, even though they are matrices (just by doing so entry by entry):

$$a_0\mathbb{I} = -AS_0\,,\ \ a_1\mathbb{I} = S_0 - AS_1\,,\ \ a_2\mathbb{I} = S_1 - AS_2\,,\ \ldots,$$

$$a_{n-1}\mathbb{I} = S_{n-2} - AS_{n-1}\,,\ \ a_n\mathbb{I} = S_{n-1}.$$

To show $f_A(A) = 0$, write

$$\begin{aligned}
f_A(A) &= f_A(A)\mathbb{I} = a_0\mathbb{I} + a_1A\mathbb{I} + a_2A^2\mathbb{I} + \ldots + a_nA^n\mathbb{I} \\
&= a_0\mathbb{I} + A(a_1\mathbb{I}) + A^2(a_2\mathbb{I}) + \ldots + A^{n-1}(a_{n-1}\mathbb{I}) + A^n(a_n\mathbb{I}) \\
&= -AS_0 + A(S_0 - AS_1) + A^2(S_1 - AS_2) + \ldots \\
&\qquad\qquad + A^{n-1}(S_{n-2} - AS_{n-1}) + A^nS_{n-1} \\
&= -AS_0 + AS_0 - AS_1 + AS_1 - AS_2 + \ldots \\
&\qquad\qquad + A^{n-1}S_{n-2} - A^nS_{n-1} + A^nS_{n-1} \\
&= 0.
\end{aligned}$$

□

SECOND PROOF OF VI.A.1. Here's a more abstract approach.

Start with a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ of F^n, and a transformation $T : \mathsf{F}^n \to \mathsf{F}^n$, with $[T]_{\mathcal{B}} = A$. We show $f_A(T)$ is the zero transformation. By definition

$$T\vec{v}_i = \sum_j A_{ji}\vec{v}_j\,,$$

which we can rewrite

$$\sum_j \left(\delta_{ij}T - A_{ji}\right)\vec{v}_j = 0.$$

Set $B_{ij} = \delta_{ij}T - A_{ji}$ (or $B = T\mathbb{I} - {}^tA$); the entries of B are formal polynomials in the transformation T. The above equation becomes

$$\sum_j B_{ij}\vec{v}_j = 0$$

while we have also

$$\det(B) = f_A(T).$$

It is therefore sufficient to show $(\det B)\vec{v}_k = 0$ for all k.

Let $\tilde{B} = \mathrm{adj} B$, so that

$$\sum_i \tilde{B}_{ki} B_{ij} = \delta_{kj} \det B.$$

This time the calculation is far less messy:

$$(\det B)\vec{v}_k = \sum_j \delta_{kj} (\det B)\vec{v}_j = \sum_j \left(\sum_i \tilde{B}_{ki} B_{ij} \right) \vec{v}_j$$

$$= \sum_{i,j} \tilde{B}_{ki} B_{ij} \vec{v}_j = \sum_i \tilde{B}_{ki} \left(\sum_j B_{ij} \vec{v}_j \right) = \sum \tilde{B}_{ki} \cdot 0 = 0.$$

□

The obvious question after Cayley-Hamilton is "can we ever do better than a polynomial of degree n?", i.e. find a nonzero polynomial of lower degree that annihilates A.

VI.A.3. EXAMPLE. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 3 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

or $A^2 - 3A = 0$, we find that $q(x) = x^2 - 3x$ annihilates A. Notice that $x^2 - 3x = x(x-3)$ divides the characteristic polynomial $f_A(x) = x^2(x-3)$ for this A.

VI.A.4. DEFINITION. The **minimal polynomial** m_A of A is the (unique) nonzero monic[3] polynomial m of lowest possible degree, such that $m(A) = 0$. Cayley-Hamilton $\Longrightarrow$ $\deg m_A \leq n$ for $n \times n$ matrices A.

[3] *Monic* means that the coeffficient of the highest power of λ (or x) in the polynomial is 1.

Uniqueness of m_A. Let d = the lowest possible degree mentioned above. If $'m$ and m are two distinct *monic* polynomials of degree d, which annihilate A, then $'m(x) - m(x) =$

$$= \left(x^d + 'a_{d-1}x^{d-1} + \ldots + 'a_0\right) - \left(x^d + a_{d-1}x^{d-1} + \ldots + a_0\right)$$

$$= ('a_{d-1} - a_{d-1})x^{d-1} + \ldots + ('a_0 - a_0).$$

Dividing by the first nonzero $'a_i - a_i$ (reading from left to right) gives a monic polynomial annihilating A:

$$''m(A) = \frac{1}{'a_i - a_i}\left('m(A) - m(A)\right) = 0.$$

But $\deg(''m) < d$, which contradicts the definition (that is, the minimality) of d.

Further properties of m_A. Now recall that *long division* of polynomials, say of g into f, gives a quotient q and remainder r (where the remainder has degree strictly less than that of g), such that $\frac{f}{g} = q + \frac{r}{g}$. If $r = 0$ then we write $g \mid f$ (g divides f), otherwise $g \nmid f$. We may write this "division algorithm" as a polynomial equation

$$\boxed{f = gq + r,\ \deg r < \deg g.}$$

VI.A.5. PROPOSITION. *The minimal polynomial of A divides its characteristic polynomial, $m_A(\lambda) \mid f_A(\lambda)$.*

PROOF. By the division algorithm we may write

$$f_A(\lambda) = m_A(\lambda) \cdot q(\lambda) + r(\lambda),\ \ \deg r < \deg m_A.$$

But then

$$r(A) = f_A(A) - m_A(A) \cdot q(A) = 0 - 0 \cdot q(A) = 0,$$

and r annihilates A. Because its degree is less than that of m_A, r must be zero (as a polynomial) — otherwise we have contradicted minimality of m_A. Therefore $f_A = m_A \cdot q$ and we're done. $\square$

VI.A.6. REMARK. By the same proof, m_A divides any polynomial p satisfying $p(A) = 0$.

In §VI.B we shall give an algorithm for finding m_A. The idea is to perform elementary row *and column* operations (suitably defined) on $\lambda\mathbb{I} - A$ to put it in a new form:

VI.A.7. DEFINITION. A matrix $M \in M_n(\mathsf{F}[\lambda])$ (with polynomial entries) is in **normal form** iff it looks like this:

$$\begin{pmatrix} f_1(\lambda) & & 0 \\ & \ddots & \\ 0 & & f_n(\lambda) \end{pmatrix}$$

where $f_1 \mid f_2 \mid \ldots \mid f_n$ and each f_i is a monic polynomial or zero.[4] (So the only possible nonzero scalar is 1, and in the sorts of normal forms we'll encounter the first few $\{f_i\}$ will usually be 1.) A typical example is $\mathrm{diag}\{1, 1, 1, \lambda, \lambda(\lambda-2)^2\}$.

In fact we shall give an algorithm associating to any square matrix M with entries in $\mathsf{F}[\lambda]$, a matrix $nf(M)$ in normal form.

We need one more

VI.A.8. DEFINITION. For $M \in M_n(\mathsf{F}[\lambda])$, let
- $\delta_k(M) :=$ the monic *gcd* (= greatest common divisor) of the determinants of all $k \times k$ submatrices[5] of M, and
- $\Delta_k(M) := \delta_k(M)/\delta_{k-1}(M)$. These Δ_k are called the **invariant factors** of M.

VI.A.9. REMARK. The $\Delta_k(M)$ are polynomials (as will be implied by the Theorem below). Note that

$$\delta_{n-1} = \text{the monic } gcd \text{ of the entries of } \mathrm{adj}(M),$$

and

$$\delta_n(M) = C^{-1} \cdot \det(M)$$

[4]The "zero" possibility will not occur when $M = nf(\lambda\mathbb{I} - A)$ (the main application), but must be included to state more general results. Note that if $f_k = 0$, then $f_{k+1} = \cdots = f_n = 0$ as well, as 0 only divides 0.

[5]The submatrices are obtained by blocking out any $(n-k)$ rows and $(n-k)$ columns. Their determinants are frequently called $k \times k$ *minors*. If these are all zero, we put $\delta_k = 0 = \Delta_k$; but again, this cannot happen for $M = \lambda\mathbb{I} - A$.

where C is a scalar – namely, the coefficient of the highest power of λ in $\det(M)$. If $M = \lambda\mathbb{I} - A$ then $\det(M)$ is monic; thus $C = 1$ and

$$\delta_n(\lambda\mathbb{I} - A) = \det(\lambda\mathbb{I} - A) = f_A(\lambda).$$

VI.A.10. THEOREM. *If*

$$nf(M) = \begin{pmatrix} f_1(\lambda) & & \\ & \ddots & \\ & & f_n(\lambda) \end{pmatrix}$$

then $\Delta_k(M) = f_k(\lambda)$. That is, the invariant factors of M are given by the diagonal entries of $nf(M)$.

The Theorem will be proved in the next section.

Clearly then $f_1(\lambda) \cdot \cdots \cdot f_n(\lambda) =$

$$\Delta_1(M) \cdot \cdots \cdot \Delta_n(M) = \delta_1(M) \cdot \frac{\delta_2(M)}{\delta_1(M)} \cdot \cdots \cdot \frac{\delta_n(M)}{\delta_{n-1}(M)} = \delta_n(M)$$

and we have a

VI.A.11. COROLLARY. *If $M = \lambda\mathbb{I} - A$ then $f_1(\lambda) \cdot \cdots \cdot f_n(\lambda) = \det(\lambda\mathbb{I} - A)$. That is, the product of the (diagonal) entries of $nf(\lambda\mathbb{I} - A)$ is $f_A(\lambda)$.*

Set

$$\delta_A(\lambda) := \delta_{n-1}(\lambda\mathbb{I} - A) = \text{monic } gcd \text{ of entries of } \mathrm{adj}(\lambda\mathbb{I} - A).$$

What we would like now is to prove the following

VI.A.12. PROPOSITION. *The top invariant factor of $\lambda\mathbb{I} - A$ is the minimal polynomial of A:*

$$m_A(\lambda) = \Delta_n(\lambda\mathbb{I} - A) = \frac{f_A(\lambda)}{\delta_A(\lambda)}.$$

According to this statement, in order to find $m_A(\lambda)$ it suffices to row/column-reduce $\lambda\mathbb{I} - A$ to normal form (as described in the next section), and pick out the last (diagonal) entry. The proof will be independent of Theorem VI.A.10.

For small n, it can actually be practical to apply the Proposition directly to compute m_A. For A as in Example VI.A.3, the entries of $\mathrm{adj}(\lambda\mathbb{I} - A)$ are all $\lambda^2 - 2\lambda$ or λ, whose gcd is λ. Dividing $f_A(\lambda) = \lambda^2(\lambda - 3)$ by this gives $m_A(\lambda) = \lambda(\lambda - 3)$.

PROOF OF PROP. VI.A.12 (IN FOUR STEPS).

$\boxed{\text{Step I}}$ Show $f_A(\lambda)/\delta_A(\lambda)$ is a polynomial (that is, $\delta_A \mid f_A$).
Let

$$B := \mathrm{adj}(\lambda\mathbb{I} - A) = \delta_A(\lambda) M \tag{VI.A.13}$$

where the *gcd* of the entries of M is 1 (see definition of $\delta_A(\lambda)$ above). By "Cramer's rule" (cf. Exercise (5) below) we know the adjoint gives a "partial" inverse to $(\lambda\mathbb{I} - A)$, i.e.

$$\det(\lambda\mathbb{I} - A)\mathbb{I} = (\lambda\mathbb{I} - A)B$$

or (using (VI.A.13))

$$f_A(\lambda)\mathbb{I} = \delta_A(\lambda)(\lambda\mathbb{I} - A)M. \tag{VI.A.14}$$

So $(\lambda\mathbb{I} - A)M$ must be of the form $\Delta(\lambda) \cdot \mathbb{I}$ (for some polynomial Δ), and the polynomial equation

$$f_A(\lambda) = \delta_A(\lambda)\Delta(\lambda)$$

must hold, and we have finished the first step. (Notice we have proved directly that $\Delta_n(\lambda\mathbb{I} - A)$ $[= \Delta(\lambda)]$ is a polynomial.)

$\boxed{\text{Step II}}$ Show $m_A(\lambda) \mid \Delta(\lambda)$.
From (VI.A.14) we have that

$$(\delta_A(\lambda)\Delta(\lambda))\,\mathbb{I} = \delta_A(\lambda)(\lambda\mathbb{I} - A)M$$

or

$$\Delta(\lambda)\mathbb{I} = (\lambda\mathbb{I} - A)M. \tag{VI.A.15}$$

Now while one cannot simply substitute A for λ (the entries of M are polynomials in λ too!), one may essentially repeat the argument we used in our first proof of Cayley-Hamilton (writing out $\Delta(\lambda) =$

$\sum b_j\lambda^j$ and $M = \sum \lambda^k B_k$) to show that

$$\Delta(A) = 0.$$

But then by Remark VI.A.6 above, m_A divides any polynomial with this property, and we are done.

$\boxed{\text{Step III}}$ <u>Show $\Delta(\lambda) \mid m_A(\lambda)$.</u>
Since by definition

$$m_A(A) = 0,$$

we have for some matrix $Q \in M_n(\mathsf{F}[\lambda])$

$$m_A(\lambda)\mathbb{I} \;=\; Q\,(\lambda\mathbb{I} - A).$$

(See Remark VI.A.16.) Multiplying on the right by M and using (VI.A.15) gives

$$m_A(\lambda)M \;=\; \Delta(\lambda)Q.$$

Consider the monic *gcd*'s of the entries of the matrices on either side:

$$m_A(\lambda) = \Delta(\lambda) \cdot gcd\{\text{entries of } Q\},$$

since $gcd\{\text{entries of } M\}$ was 1. This concludes step III.

$\boxed{\text{Step IV}}$ <u>The end</u>.
Since m_A and Δ are both monic (Δ is the quotient of two monic polynomials), and both divide each other, they must be equal. □

VI.A.16. REMARK. How do we know that $m_A(A) = 0$ means that $m_A(\lambda)\mathbb{I}$ is "divisible" by $(\lambda\mathbb{I} - A)$ in a matrix ring which is not even commutative? The trick is to look just at the (commutative) subring R consisting of polynomials in A and λ. If you are comfortable with rings, then consider the homomorphism $\theta : R \twoheadrightarrow R/(\lambda - A)$, with kernel simply the ideal $(\lambda - A)$ consisting of multiples of $\lambda - A$. In the quotient, λ is identified with A; and so $\theta(m_A(\lambda)) = \theta(m_A(A)) = 0$. Consequently $m_A(\lambda)$ is a multiple of $\lambda - A$, as required.

Alternatively, writing $m_A(\lambda)\mathbb{I} = \lambda^\mu\mathbb{I} + \sum_{k=0}^{\mu-1} \lambda^k\alpha_k\mathbb{I}$ and

$$Q(\lambda\mathbb{I} - A) = \textstyle\sum_{k=0}^{\mu-1} \lambda^k Q_k(\lambda\mathbb{I} - A),$$

with $Q_k \in M_n(\mathsf{F})$, we can try to solve for Q_k such that these two expressions are equal. One finds that $Q_{\mu-1} = \mathbb{I}$, $Q_{\mu-2} = \alpha_{\mu-1}\mathbb{I} + A$, $Q_{\mu-3} = \alpha_{\mu-2}\mathbb{I} + \alpha_{\mu-1}A + A^2, \ldots,$ and

$$Q_0 = \alpha_1\mathbb{I} + \alpha_2 A + \cdots + \alpha_{\mu-1}A^{\mu-2} + A^{\mu-1}.$$

This leaves the equality of the constant terms, which is

$$\alpha_0\mathbb{I} = -Q_0 A.$$

As you will readily verify, this is just the statement that $m_A(A) = 0$.

Exercises

(1) Verify Cayley-Hamilton for

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then use Prop. VI.A.12 and Defn. VI.A.8 to directly find m_A.

(2) Same as the last Exercise, for

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

(3) Suppose $\det A \neq 0$. Use Cayley-Hamilton to show that A is invertible and that A^{-1} is given by a certain polynomial in A.

(4) Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial, and V be a vector space with basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$. Define $T \colon V \to V$ by $T\vec{v}_i = \vec{v}_{i+1}$ ($i = 1, \ldots, n-1$) and

$$T\vec{v}_n = -a_{n-1}\vec{v}_n - a_{n-2}\vec{v}_{n-1} - \cdots - a_1\vec{v}_2 - a_0\vec{v}_1.$$

(a) Show that $p(T) = 0$. [Hint: substitute $\vec{v}_2 = T\vec{v}_1$, etc., into the equation for $T\vec{v}_n$.]

(b) Determine $A := [T]_{\mathcal{B}}$.

(c) Show that $p(x) = m_A(x) = f_A(x)$. [Hint: suppose that $q(T) = 0$ for a polynomial q of degree $< n$.]

(5) Prove that for a matrix with entries in $F[\lambda]$ (or really, any commutative ring), we have

$$M \cdot \mathrm{adj}(M) = \det(M)\mathbb{I} = \mathrm{adj}(M) \cdot M.$$

[Hint: all you need is the fact that by definition, $[\mathrm{adj}(M)]_{ij} = (-1)^{i+j} \det(M_{\widehat{ji}})$, together with the Laplace expansion formulas for det and the property of det that a repeated row makes it zero.[6] The point is to *not* use Cramer's rule.]

[6]Both of these follow from the definition (IV.A.5) of det for matrices with coefficients in any commutative ring, like $\mathsf{F}[\lambda]$.