VII.F. Fourier series

In this final section, I outline how some of these ideas extend to an infinite-dimensional inner product space.[35] Let

$$\mathbb{V} \;:=\; \text{square-integrable functions on the unit circle,}$$

by which we mean real-valued, measurable functions f on $\mathbb{R}$ with $f(x+2\pi) = f(x)$ and $\int_0^{2\pi} f(x)^2 dx < \infty$. (Think of x as "angle" on the circle.) We equip $\mathbb{V}$ with the L^2 inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx.$$

VII.F.1. FACT. *The collection*

$$\mathcal{B} := \left\{\tfrac{1}{\sqrt{\pi}}\cos kx\right\}_{k\in\mathbb{N}} \cup \left\{\tfrac{1}{\sqrt{\pi}}\sin kx\right\}_{k\in\mathbb{N}} \cup \left\{\tfrac{1}{\sqrt{2\pi}}\right\}$$

is an orthonormal basis for $\mathbb{V}$.

We won't prove that $\mathcal{B}$ "spans" $\mathbb{V}$ — that's a topic for a course in functional analysis — but here is how to do orthonormality (and thus independence): use the trigonometric identities

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

to build "product formulas", e.g.

$$\cos mx \cos nx \;=\; \frac{1}{2}\left\{\cos[(m-n)x] + \cos[(m+n)x]\right\}.$$

Then integrate to obtain (m, $n \geq 0$)

$$\int_0^{2\pi} \cos(mx)\cos(nx)dx \;=\; \pi\delta_{mn} = \int_0^{2\pi} \sin(mx)\sin(nx)dx,$$

$$\int_0^{2\pi} \cos(mx)\sin(nx)dx \;=\; 0.$$

[35] Technically, one should also ask for the space to be *complete* with respect to the metric given by the inner product — that is, for it to be a *Hilbert space*. (This is true for $\mathbb{V}$.)

Recall that for a finite-dimensional inner product space V with orthonormal basis $\mathcal{B} = \{\hat{v}_1, \ldots, \hat{v}_n\}$, the "Fourier expansion formula" for $\vec{x} \in \mathbb{V}$ reads

$$\vec{x} = \sum_{i=1}^{n} \langle \hat{v}_i, \vec{x} \rangle \, \hat{v}_i.$$

Similarly, for $f \in P$, we might expect[36]

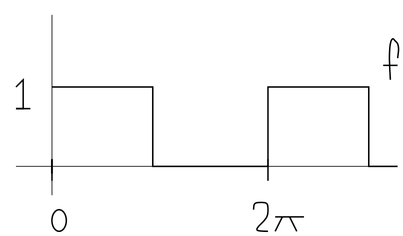
$$\begin{aligned} f(x) = \sum_{k=1}^{\infty} \Bigg\{ &\left(\tfrac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \cos(kx) dx \right) \tfrac{1}{\sqrt{\pi}} \cos kx \\ &+ \left(\tfrac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \sin(kx) dx \right) \tfrac{1}{\sqrt{\pi}} \sin kx \Bigg\} + \left(\tfrac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) dx \right) \tfrac{1}{\sqrt{2\pi}} \\ = \; & \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right) \; + c \end{aligned}$$

where

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) dx, \;\; b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx,$$
$$c = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$$

are called *Fourier coefficients*.

VII.F.2. EXAMPLE. Consider the "binary oscillation" function



For this f, we have immediately $c = \frac{1}{2}$, and $a_k = 0$ by symmetry (since f is odd). Moreover

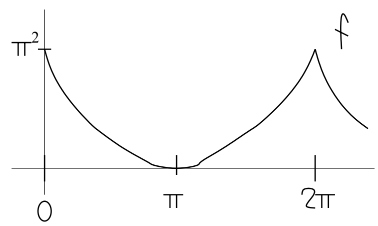
$$\begin{aligned} b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx \; &= \; \frac{1}{\pi} \int_0^{\pi} \sin(kx) dx \\ &= -\frac{1}{\pi k} \cos(kx) \Big|_0^{\pi} = \begin{cases} 0, & k \text{ even} \\ \frac{2}{\pi k}, & k \text{ odd} \end{cases}, \end{aligned}$$

[36]See the remarks at the end of the section.

and so

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{\substack{k>0 \\ \text{odd}}} \frac{1}{k} \sin(kx).$$

VII.F.3. EXAMPLE. $f(x) = (x-\pi)^2$ on $[0, 2\pi]$ (repeated periodically):



This time we have by symmetry $b_k = 0$, and

$$c \;=\; \frac{1}{2\pi} \int_0^{2\pi} (x-\pi)^2 dx \;=\; \frac{\pi^2}{3}.$$

Using even symmetry of $\cos(kx)$ about π, then integration by parts twice, we have

$$\begin{aligned} a_k =\; & \frac{1}{\pi} \int_0^{2\pi} (x-\pi)^2 \cos(kx) dx \;=\; \frac{2}{\pi} \int_0^{\pi} (x-\pi)^2 \cos(kx) dx \\ =\; & \frac{2}{\pi} \left[\underbrace{\frac{(x-\pi)^2}{k} \sin(kx) \Big|_0^{\pi}}_{0} - \frac{2}{k} \int_0^{\pi} (x-\pi) \sin(kx) dx \right] \\ =\; & -\frac{4}{\pi k} \int_0^{\pi} (x-\pi) \sin(kx) dx \\ =\; & \frac{4}{\pi k} \left[\frac{(x-\pi)}{k} \cos(kx) \Big|_0^{\pi} - \underbrace{\frac{1}{k} \int_0^{\pi} \cos(kx) dx}_{0} \right] \\ =\; & \frac{4}{k^2}. \end{aligned}$$

Therefore

$$f(x) \;=\; \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} \cos(kx).$$

Notice that by evaluating at 0 we have immediately

$$\pi^2 = f(0) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} \quad \longrightarrow \quad \frac{2\pi^2}{3} = \sum_{k=1}^{\infty} \frac{4}{k^2}$$

which yields Euler's famous formula

$$\boxed{\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}}.$$

Heat equation on a circle. Interpreting the function of Example VII.F.2 as the heat distribution on a circular piece of iron that has been held halfway into the fire, we ask: how does this distribution behave over time once the iron is removed from the fire? Somewhat unrealistically, we will assume no outside influence after this point: the iron is a closed 1-dimensional system.

To answer this question, we shall find (in terms of Fourier series) the solution $f(x,t)$ to the heat equation

$$\text{(VII.F.4)} \qquad \frac{\partial f}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 f}{\partial x}(x,t) \ \ [\, + \underset{\substack{\uparrow \\ \text{heat source}}}{h(x,t)}\,]$$

with initial condition $f(x,0) = f(x) \in \mathbb{V}$. In particular, f is periodic in x with period 2π. The "closed system" assumption means that $h \equiv 0$ (at least for $t \geq 0$, where we are solving the equation).

We first indicate how to "derive" (VII.F.4). Start by writing an equation that says

$$\begin{array}{c} \text{rate of change of total heat} \\ \text{stored in } [x,\, x+\Delta x] \end{array} = \begin{array}{c} \text{rate of heat flux thru } x \text{ and} \\ x+\Delta x\,, \text{ into } [x,\, x+\Delta x] \end{array}$$

(since we assume there is no external heat source). Now it should make sense that the heat flux into $[x,\, x+\Delta x]$ at $x+\Delta x$ is proportional to the *slope* $\frac{\partial f}{\partial x}(x+\Delta x,\, t)$. Writing α^2 for a proportionality constant,[37] our equation is

$$\text{(VII.F.5)} \qquad \begin{array}{c} \displaystyle \alpha^{-2}\frac{d}{dt}\int_x^{x+\Delta x} f(x,t)\,dx \\ \| \\ \displaystyle \alpha^{-2}\int_x^{x+\Delta x} \frac{\partial f}{\partial t}(x,t)\,dx \end{array} = \frac{\partial f}{\partial x}(x+\Delta x,\, t) - \frac{\partial f}{\partial x}(x,t).$$

[37] I write α^2 so you don't forget it's positive.

According to the Mean-Value Theorem, for $g(x)$ continuously differentiable on $[x,\, x+\Delta x]$, there exists $\xi \in (x,\, x+\Delta x)$ such that

$$g(x+\Delta x) - g(x) = g'(\xi)\Delta x.$$

Applying this to $\frac{\partial f}{\partial x}(x,t) =: g_t(x)$ for each fixed[38] t gives

$$\frac{\partial f}{\partial x}(x+\Delta x,\, t) - \frac{\partial f}{\partial x}(x,t) = \frac{\partial^2 f}{\partial x^2}(\xi,t)\,\Delta x, \tag{VII.F.6}$$

and combining (VII.F.6) with (VII.F.5) yields

$$\alpha^{-2}\int_x^{x+\Delta x} \frac{\partial f}{\partial t}(x,t)\,dx \;=\; \frac{\partial^2 f}{\partial x^2}(\xi,t)\,\Delta x. \tag{VII.F.7}$$

Dividing both sides of (VII.F.7) by Δx and taking the limit as $\Delta x \to 0$ ($\implies \xi \to x$), we recover the heat equation (VII.F.4) (with $h=0$).

Now remember how, in the finite-dimensional setting, we solved continuous dynamical systems of the form

$$\frac{d\vec{X}}{dt} = A\vec{X}$$

when $A \in M_n(\mathbb{R})$ is diagonalizable over $\mathbb{R}$. If $\vec{X}(0) \in \mathbb{R}^n$ is an eigenvector with eigenvalue λ the solution is just $e^{\lambda t}\vec{X}(0)$. More generally, if $\{\vec{v}_1,\ldots,\vec{v}_n\} \subset \mathbb{R}^n$ is an A-eigenbasis (with eigenvalues $\lambda_1,\ldots,\lambda_n$), we can always write $\vec{X}(0) = \sum_i c_i\vec{v}_i$ and then the solution is

$$\vec{X}(t) \;=\; \sum_i e^{\lambda_i t}\, c_i\, \vec{v}_i.$$

Replacing vectors by functions and A by $T = \alpha^2\frac{d^2}{dx^2}$, notice that

$$T\left(\frac{1}{\sqrt{\pi}}\cos kx\right) \;=\; \frac{\alpha^2}{\sqrt{\pi}}\frac{d^2}{dx^2}\cos kx \;=\; -\alpha^2k^2\left(\frac{1}{\sqrt{\pi}}\cos kx\right);$$

in fact, the eigenfunctions of T are just the elements of our basis $\mathcal{B} = \left\{\frac{1}{\sqrt{2\pi}},\, \frac{1}{\sqrt{\pi}}\cos kx,\, \frac{1}{\sqrt{\pi}}\sin kx\right\}_{k\in\mathbb{N}}$, with respective eigenvalues $0,\, -\alpha^2k^2,\, -\alpha^2k^2$. Therefore if

$$f(x,0) \;=\; \sum_{k=1}^{\infty}\left(a_k\cos kx + b_k\sin kx\right) \;+ C$$

[38]Note in particular that $g_t'(\xi)$ is just $\frac{\partial^2 f}{\partial x^2}(\xi,t)$.

we have for

$$\frac{\partial f}{\partial t} = Tf$$

the solution

$$f(x,t) \;=\; \sum_{k=1}^{\infty} e^{-\alpha^2 k^2 t} \left(a_k \cos kx + b_k \sin kx \right) \; + C.$$

Finally, for $f(x,0) =$ the function of Example VII.F.2,

$$f(x,t) \;=\; \frac{1}{2} + \frac{2}{\pi} \sum_{\substack{k>0 \\ \text{odd}}} \frac{e^{-\alpha^2 k^2 t}}{k} \sin kx.$$

Notice that the highest frequencies (large k) are suppressed the most quickly, so that the sharp corners disappear. As a consequence the function gets "smoothed" over time by heat conduction, eventually approaching the average heat value $\frac{1}{2}$.

Remarks on convergence. The convergence of Fourier series may be considered in several different senses, including (L^2-)norm convergence, pointwise convergence, and uniform convergence. The first of these means that $\|f - S_N\| \to 0$ as $N \to \infty$, where S_N are the N^{th} partial sums of the series. This is true quite generally for square-integrable functions, and is the sense in which $\mathcal{B}$ spans $\mathbb{V}$.

More relevant to our purposes above, however, is pointwise convergence $S_N(x_0) \to f(x_0)$. Continuity is not the right condition here — it only ensures pointwise convergence *almost everywhere* — and excludes the rectangular wavefunction of Example VII.F.2. Let's call a function *piecewise continuous* if it has only finitely many discontinuities, and moreover possesses (finite) left and right limits at each discontinuity. (Such a function is necessarily bounded.) Then if f *and* its first derivative are piecewise continuous, the Fourier series converges pointwise to the function $\tilde{f}$, which is given by f away from f's discontinuities, and by the average of left and right limits at f's discontinuities. This is a bit more than needed, but is clearly suitable for both of the examples above.

The idea that a discontinuous function like that in Example VII.F.2 should be representable by a trigonometric series caused controversy and led to the rejection of Fourier's 1807 paper. Eventually he became president of the professional society responsible for this rejection and had it published in their prestigious journal. With this small reminder of the value of persistence, I congratulate the reader for persevering to the end. Now try some exercises:

Exercises

(1) Find a series formula for $\frac{\pi}{4}$ by evaluating the result of Example VII.F.2 at $x = \frac{\pi}{2}$.

(2) Can you cook up an example to compute $\sum_{k=1}^{\infty} \frac{1}{k^4}$?

(3) Let $P^{\infty} \subseteq P$ denote the subspace of smooth (infinitely differentiable) functions. Show that $i\frac{d}{dx}$ is self-adjoint in the inner product defined above. What are its eigenvectors and eigenvalues?

(4) Let V denote the vector space of smooth complex-valued functions on $\mathbb{R}$ with "rapid (exponential) decay at $\pm\infty$". (We won't need to make this precise.) The inner product is

$$\langle f, g \rangle := \int_{-\infty}^{\infty} \overline{f(x)} g(x) dx.$$

(a) Compute the adjoint of $A := \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right)$, $AA^{\dagger}$ and $A^{\dagger}A$. Conclude that $H := \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2 \right)$ is self-adjoint; it turns out that this is the *Schrödinger Hamiltonian* of the "quantum harmonic oscillator".

(b) Define $\psi_0 := e^{-x^2/2}$ and (for $n \in \mathbb{N}$) $\psi_n := A^n \psi_0$. Show that $\psi_n(x) = e^{-x^2/2} h_n(x)$ for some polynomials h_n: these are the *Hermite polynomials*; compute them explicitly for a few values of n.

(c) Prove that the ψ_n are eigenfunctions for the Hamiltonian, and determine the eigenvalues. (These correspond to quantum states of the system, with energies proportional to the eigenvalues.) [Hint: first show (e.g. using induction) that $A^{\dagger}A^{n+1} - A^{n+1}A^{\dagger} = nA$, then check that $A^{\dagger}$ kills ψ_0, and finally use the relation between $A^{\dagger}A$ and H from part (a).]