Problem set 1

(1) Let \( \{f_\alpha\} \) be a normal family of holomorphic functions on a region \( U \). Show that \( \{f_\alpha'\} \) is a normal family.

(2) Let \( \beta \in D_1 \) and \( f(z) = \frac{z-\beta}{1-\beta z} \). Prove that the sequence \( \{f_n\} \) defined by \( f_1 = f \), \( f_{n+1} = f \circ f_n \) converges normally, and find the limit function. [Hint: use (1); OK to do for a subsequence.]

(3) Let \( \Omega \subset \mathbb{C} \) be as in the Riemann mapping theorem, with the additional assumption that \( \Omega \) be symmetric with respect to the real axis. Let \( f : \Omega \to \mathbb{C} \) be a conformal isomorphism sending \( p \in \Omega \) to 0, with \( p \in \mathbb{R} \) and \( f'(p) \in \mathbb{R}_+ \). Prove that \( \overline{f(z)} = f(z) \). [Hint: use the uniqueness part of the RMT]

(4) Let \( U \subset \mathbb{C} \) be a bounded region, \( \{f_j\} \subset \text{Hol}(U) \) a sequence with \( \int_U |f_j(z)|^2 dx \, dy < C < \infty \) (where \( C \) is independent of \( j \)). Prove that \( \{f_j\} \) is a normal family. [Hint: for \( z_0 \in D(z_0, \epsilon) \subset U \), use the Cauchy integral formula to bound \( |f_j(z_0)|^2 \) by \( C/\pi \epsilon^2 \); then show the \( \{f_j\} \) are locally uniformly bounded and use Lecture 1.]

(5) Let \( U \) be a region and \( \mathfrak{F} = \{f_\alpha\} \subset \text{Hol}(U) \) a family with \( \text{Re}(f_\alpha) > 0 \) on \( U \). Prove that \( \mathfrak{F} \) s normal in the classical sense: any sequence \( \{f_n\} \subset \mathfrak{F} \) contains a subsequence converging uniformly on compact sets OR tending uniformly to \( \infty \) on compact sets. [Hint: consider \( e^{-f} \), and use Hurwitz.]