Problem set 8

Let \( \omega_1, \omega_2 \) be two complex numbers, linearly independent over \( \mathbb{R} \), and \( \Lambda = \mathbb{Z} \langle \omega_1, \omega_2 \rangle \subset \mathbb{C} \) be the lattice they generate.

1. Show that the series \( \frac{1}{z^2} + \sum_{\omega \in \Lambda} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \) defining \( \varphi(z) \) is absolutely and uniformly convergent on any compact subset of \( \mathbb{C} \) which does not contain any of the points of \( \Lambda \). Try to do this without the Lemma on p. 1 of Lecture 15 (perform a direct estimate).

2. There are three perspectives on the addition theorem for \( \wp \): analytic, geometric, and algebraic. I’ll do the latter two in class and you’ll provide the analytic approach here, by following the steps below. Consult Ahlfors pp. 276-7 for hints.
   
   (a) \( \varphi(z) - \varphi(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} \)
    
   (b) \( \frac{\varphi'(z)}{\varphi(z)-\varphi(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z) \)
    
   (c) \( \zeta(z+u) = \zeta(z) + \zeta(u) + \frac{1}{2} \frac{\varphi'(z)-\varphi'(u)}{\varphi(z)-\varphi(u)} \)
    
   (d) \( \varphi(z+u) = -\varphi(z) - \varphi(u) + \frac{1}{4} \left( \frac{\varphi'(z)-\varphi'(u)}{\varphi(z)-\varphi(u)} \right)^2 \) [addition formula]
    
   (e) \( \varphi(2z) = \frac{1}{4} \left( \frac{\varphi'(z)}{\varphi(z)} \right)^2 - 2\varphi(z) \)

3. For this problem, assume that \( \Lambda \) such that \( g_2 = -4 \) and \( g_3 = 0 \).
   (That is, \( \mathbb{C}/\Lambda \) is isomorphic to the curve \( y^2 = 4x^3 + 4x \); such a \( \Lambda \) exists simply by taking it to be the set of all periods of \( dx/y \) on this curve.)
   
   (a) Express the RHS of formula 2(e) as a rational function of \( \varphi(z) \).
    
   (b) Let \( u \) be such that \( \varphi(u) = 1 \) (and \( \varphi'(u) = 2\sqrt{2} \)). Show that \( u \) is 4-torsion, i.e. \( 4u \in \Lambda \).
    
   (c) Let \( u \) be such that \( \varphi(u) = \frac{p}{2^aq} \), where the fraction is written in lowest terms, \( a \) is an odd natural number and \( p \) and \( q \) are odd integers. Show that \( u \) is “of infinite order”, i.e. no integer multiple of it lies in \( \Lambda \). [Hint: put \( u_0 := u \). Show that \( \varphi(u_1) \), where \( u_1 = 2u_0 \), is of the same form, with bigger \( a \), and iterate. Then suppose \( u_0 \) was \( N \)-torsion and produce a contradiction via the pigeonhole principle.]
(4) This exercise concerns the Theta function, defined in Lecture 14. (a) Check that $\theta(z+1) = \theta(z)$. (b) Check that $\theta(z) = Ce^{h(z)}\sigma(z + \frac{\tau + 1}{2})$, where $h(z) = -\frac{\eta_1}{2}z^2 - (\frac{\eta_1 \tau}{2} + \frac{\eta_1}{2} + \pi i)z$. [See p. 2 of Lecture 15]