

PROBLEM SET 5

Page numbers refer to my notes. In addition to these exercises, you should look at the two exercises at the end of Chapter 6 in Voisin. (If you're feeling particularly adventurous, you could also try some of those at the end of Chapter 6 in Warner.)

- (1) Verify that  $\partial/\partial\bar{z}$  is an elliptic operator on  $\mathbb{C}$ -valued functions on  $U \subset \mathbb{C}$ .
- (2) (i) Check that  $\Delta_d$  commutes with Hodge  $*$ . (ii) Given  $\alpha \in A^r(M)$  ( $M$  compact Riemannian manifold) closed, show that  $*\alpha$  is closed if and only if  $\alpha$  is harmonic. (iii) If  $M$  is complex Hermitian, prove that  $*$  takes forms of bidegree  $(p, q)$  to forms of bidegree  $(n - q, n - p)$ .
- (3) Consider the Heisenberg group  $M = \left\{ \left( \begin{array}{ccc} 1 & x & y \\ & 1 & z \\ & & 1 \end{array} \right) \middle| x, y, z \in \mathbb{C} \right\} \subset GL_3(\mathbb{C})$  and put  $\Gamma := M \cap GL_3(\mathbb{Z})$ . Show that the Iwasawa manifold  $\Gamma \backslash M$  is non-Kähler by exhibiting a non-closed holomorphic form. [Hint: consider " $M^{-1}dM$ ".]
- (4) Let  $L \rightarrow X$  be a holomorphic line bundle over a compact complex  $n$ -manifold, such that for some  $N > 0$   $H^0(X, \mathcal{O}(L^{\otimes N})) \neq 0$ . Prove that if also  $H^n(X, K_X \otimes \mathcal{O}(L)) \neq 0$ , then  $L$  is trivial.
- (5) Prove the pairings  $Q_k$  (cf. p. 156) are well-defined.
- (6) Check the well-definedness of Poincaré residue (i.e. independence of choice of local coordinates).
- (7) Compute  $h^{1,1}$  for quintic surfaces in  $\mathbb{P}^3$  and  $h^{2,1}$  for quintic threefolds in  $\mathbb{P}^4$ .
- (8) Work out the constants in powers of Laurent polynomials and check the Picard-Fuchs equation (F.11) on pp. 175-6, in the special case  $n = 2$ . This is a degree 3 ODE; in fact one can do better: find a degree 2 differential equation satisfied by  $\mathfrak{P}(t)$ . Then find the first few terms of a solution of the form  $\log(t)\mathfrak{P}(t) + \mathfrak{Q}(t)$  (where  $\mathfrak{Q}$  is a power series about  $t = 0$  like  $\mathfrak{P}$ ).