Algebraic cycles and representation theory

PART I : Algebraic cycles and their Hodge-theoretic invariants

A. Cycle groups

1. Operations on cycles

Let $k = \bar{k}$ be a field (algebraically closed), X/k a smooth quasi-projective variety of dimension d. We define the groups of algebraic cycles:

<u>Definition 1</u>: $Z_q(X) = Z^{d-q}(X) :=$ the free abelian group generated by subvarieties $W \subset X$ (irred/k) of dim. q (cd. $d-q$)

$$\underset{\displaystyle Z = \sum m_i V_i \ (m_i \in \mathbb{Z})}{\in}$$

Given a morphism $f: X \to Y$ of varieties, we define the <u>push-forward</u> of cycles by

$$(1) \qquad f_* : Z_q(X) \to Z_q(Y)$$

$$\underset{(\text{irred})}{W} \longmapsto \begin{cases} 0 & \text{if } \dim \overline{f(W)} < \dim W \\ [k(W) : k(\overline{f(W)})] \cdot \overline{f(W)} & \text{if } \dim \overline{f(W)} = \dim W \end{cases}$$

(Zariski closure)

This preserves <u>dimension</u>.

Next, given two cycles Z_1 and Z_2, we want to define an <u>intersection product</u>. We define on the level of irreducible subvarieties $V, W \subset X$ of cd. i resp. j, then extend by linearity.

First write $V \cap W = \cup V_\ell$, V_ℓ irred. of cd. $\le i+j$.

This intersection is <u>proper</u> $\overset{\text{def.}}{\iff}$ $\mathrm{cd}_X V_\ell = i+j$ $(\forall \ell)$, in which case

$$(2) \quad i(V_\ell\,;\, V\cdot W\,;\, X) := \sum_{r=0}^{d} (-1)^r \underset{\substack{\uparrow \\ \text{length}}}{\ell_{\mathcal{O}}} \left\{ \mathrm{Tor}_r^{\mathcal{O}}\left(\mathcal{O}/I_V\,,\, \mathcal{O}/I_W\right)\right\} \qquad [\mathcal{O} = \mathcal{O}_{V_\ell, X}]$$

and

$$(3) \quad V\cdot W := \sum i(V_\ell\,;\, V\cdot W\,;\, X)\, V_\ell \in Z^{i+j}(X).$$

There is a more geometric way to understand the intersection multiplicities (2) : given a fiber square

$$\begin{tikzcd} \mathfrak{z} \subset f^{-1}(W) = Y \arrow[r, hook] \arrow[d] & \tilde V \arrow[d, "f"] \\ W \arrow[r, hook, "\substack{\text{regular} \\ \text{emb. of codim. } j}"'] & X \end{tikzcd}$$

(irred. component of Y) ($\tilde V$ mapping onto V & of same dimension (e.g. resolution of sings.))

with V, W as above, write $I_Y \subset \mathcal{O}_{\mathfrak{z},\tilde V}$ for the ideal of Y.

The quotient has finite length and

$$(4) \quad 1 \le i(\underset{\substack{\| \\ \text{(this is one of the } V_\ell\text{'s)}}}{f(\mathfrak{z})}\,;\, V\cdot W\,;\, X) \overset{(*)}{\le} \ell(\mathcal{O}_{\mathfrak{z},\tilde V}/I_Y)$$

where $(*)$ is an <u>equality</u> if $\mathfrak{z} \hookrightarrow \tilde V$ is regular.

Ex/ $X = \mathbb{A}^4$, $\tilde V = \mathbb{A}^2$, V = image of $f : \mathbb{A}^2 \to \mathbb{A}^4$, $(s,t) \mapsto (s^4, s^3t, st^3, t^4)$, $W = V(x_1, x_4)$ $\Downarrow$ $f^{-1}W = V(s^4, t^4)$

$\mathfrak{z}$ = point $\underline{0} \in \mathbb{A}^2$, so $(*)$ is equality

$$\Rightarrow i(\underset{f(\mathfrak{z})}{\underline{0}}\,;\, V\cdot W\,;\, X) = \ell(\mathcal{O}_{\underline{0},\mathbb{A}^2}/I_{f^{-1}(W)}) = \ell(k[s,t]/(s^4,t^4))$$

$$= 16.$$

This is all in line with the general philosophy of <u>parametrizing one of the varieties being intersected</u>, & <u>pulling back the defining eqns. of the other</u> //

Now we turn to cycle-theoretic pullback along a morphism $f : X \to Y$. Write π_X, π_Y for the projections $X \times Y \rightrightarrows X, Y$, and $\Gamma_f \subset X \times Y$ for the graph of f. Given any $Z \in Z^p(Y)$ for which

(5) $$\Gamma_f \cap (X \times Z) \text{ is proper,}$$

we may define

(6) $$f^*(Z) := (\pi_X)_* \{ \Gamma_f \cdot (X \times Z) \} \in Z^p(X).$$

(Notice that this preserves codimension.) The issue with (5) is that we need Z to intersect properly the locus in Y along which the fiber-dimension of f jumps (up), as well as $\underbrace{\text{its image } \overline{f(X)}}_{\text{the Zariski closure of}}$. (*)

Proposition 1: If f is $\overset{\text{dominant +}}{\text{flat}}$, then $f^* : Z^p(Y) \to Z^p(X)$ is defined on all cycles.

Proof: For f dominant, flatness is equivalent to equidimensionality. Since (5) $\Leftrightarrow$ (*), the Prop. follows. $\square$

This is often called "flat pullback", but the criterion (*) is much more general & useful.

Remark 1:
- f dominant means $\overline{f(X)} = Y$
- f flat means that any exact sequence of $\mathcal{O}_{y,Y}$ modules remains exact upon tensoring $\otimes_{\mathcal{O}_{y,Y}} \mathcal{O}_{x,X}$ ($\forall y \in Y$, $x \in f^{-1}(y) \subset X$).

Remark 2: Chap. 8 of Fulton's Intersection Theory describes a refined intersection product which does not require proper intersection. If $\underset{\dim i}{V}$ and $\underset{\dim j}{W}$ are subvarieties of $\underset{\dim d}{X}$ (smooth), then one may define a cycle class $\widetilde{V \cdot W} \in CH_{i+j-d}(V \cap W)$ whose push-forward into $CH_{i+j-d}(X)$ is the correct class. (See §§2-3 below.) Likewise there is a "refined pullback": given $f: X \to Y$ and $Z \overset{\mathrm{cd}\, p}{\subset} Y$, this is a class $\widetilde{f^* Z} \in CH^*(f^{-1}Z)$ which maps to $f^* Z \in CH^p(X)$.