2. Equivalence relations

The algebraic cycle groups are best studied modulo some equivalence relation, so that we can always take intersection products of cycle classes, take pullbacks, etc. Assume X, Y are projective (or just proper) & smooth.

Definition 1 (Samuel): An equivalence relation "$\equiv$" on cycles is adequate if the following hold:

(RA_1) $\{Z \in Z^i(X) \mid Z \equiv 0\} < Z^i(X)$ is a subgroup $(\forall i, X)$

(RA_2) $\exists\, Z' \equiv Z$ such that $Z' \cap W$ is proper $(\forall Z \in Z^i(X),\ \underset{\text{subvar.}}{W \subset X})$

(RA_3) for any $Z \in Z^*(Y)$ and $T \in Z^*(X \times Y)$, such that $T \cap (X \times Z)$ is proper, and $\underset{(\text{on } Y)}{Z \equiv 0}$, we have

$$T(Z) := (\pi_X)_* \{(X \times Z) \cdot T\} \underset{(\text{on } X)}{\equiv 0}.$$

Proposition 1: For $\equiv$ an adequate equiv. relation

(a) $\underset{(\text{on } X')}{Z \equiv 0} \implies Z \times X'' \equiv 0$ on $X' \times X''$

(b) $\underset{(\text{on } X)}{Z \equiv 0}$ & $\underset{(W \subset X)}{Z \cap W \text{ proper}} \implies Z \cdot W \equiv 0$

(c) $\mathfrak{Z} \in Z^*(X \times Y)$, $\mathfrak{Z} \equiv 0 \implies (\pi_X)_* \mathfrak{Z} \equiv 0$

(d) $\underset{(\text{on } X)}{Z \equiv 0}$, $W \subset X$ nonsingular, $Z \cap W$ proper $\implies \underset{(\text{on } \underset{\sim}{W})}{Z \cdot W \equiv 0}$.

(e) $\underset{(\text{on } X)}{Z \equiv 0}$, $X \overset{\iota}{\subset} Y \implies \underset{(\text{on } \underset{\sim}{Y})}{(\iota_*) Z \equiv 0}$

[Note: Converse to (e) is FALSE — can have $Z \in Z^i(X)$, $X \overset{\iota}{\subset} Y$, $(\iota_*) Z \equiv 0$ on Y, $Z \not\equiv 0$ on X.]

<u>Proof of (a) & (b):</u> (a) follows from (RA_3) by taking $T = \Delta_{X'} \times X''$, $Y = X'$, $X = X' \times X''$. [$\Delta_{X'}$ is the <u>diagonal</u> in $X' \times X'$.]

(b) follows from (RA_3) by taking $T = (X \times W) \cdot \Delta_X$ on $X \times X$, so

$$\begin{array}{ccc} (Z \times W) \cdot \Delta_X & = (Z \times X) \cdot (X \times W) \cdot \Delta_X = & (Z \times X) \cdot T \\ {\scriptstyle \pi_X} \downarrow & & \\ Z \cdot W & \overline{\overline{\qquad\qquad}} & T(Z). \end{array}$$

$\square$

[Exercise: try another!]

<u>Proposition 2:</u> Write $C^i_{\equiv}(X) := Z^i(X) / Z^i_{\equiv}(X)$ for the quotient group of equivalence classes. Then

(a) $C^{\cdot}_{\equiv}(X) := \bigoplus_i C^i_{\equiv}(X)$ forms a commutative ring, graded by codimension, with identity $\langle X \rangle$.

(b) Given $T \in Z^j(X \times Y)$, $T(\cdot) : C^{\cdot}_{\equiv}(X) \to C^{\cdot}_{\equiv}(Y)$ is a homomorphism of abelian groups, depending only on the class $\langle T \rangle \in C^j(X \times Y)$. A special case is when $T = {}^t\Gamma_f$, $f : X \to Y$, $T(\cdot) = f_*$.

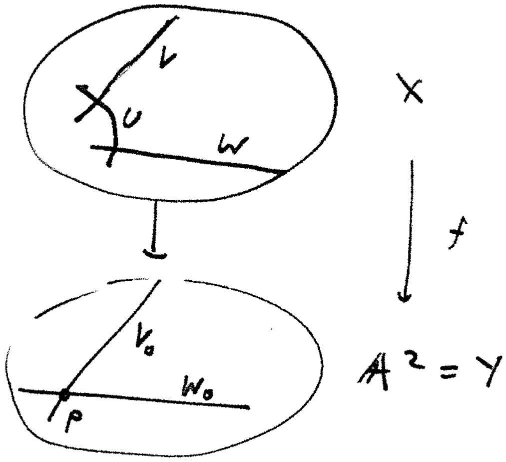
(c) Given a morphism $f : X \to Y$, $f^* : C^{\cdot}_{\equiv}(Y) \to C^{\cdot}_{\equiv}(X)$ is a homomorphism of commutative rings.

<u>Proof:</u> (Sketch) (a) use Prop. 1 (b), RA_1, RA_2

(b) use Prop. 1 (b), RA_2, RA_3

(c) take $T = \Gamma_f$, use (b); the ring assertion follows from associativity of intersection product. $\square$

<u>Remark 1</u>: In general it is <u>false</u> that f_* gives a ring homomorphism. Consider the case of a blow-up



Then $f_* V \cdot f_* W = V_0 \cdot W_0 = p$, but $f_*(V \cdot W) = f_*(0) = 0$.

(To see that $f^*(V_0 \cdot W_0) = f^*(V_0) \cdot f^*(W_0)$, you have to either use Fulton's refined product [& pullback!] or move V_0, W_0.)

<u>Remark 2</u>: We can define cycle groups for quasi-projective varieties as follows: if $X \subset \overline{X}$ is a smooth compactification, then $Z^i(X) = Z^i(\overline{X}) / Z^{i-c}(\overline{X} \setminus X)$ (where I assume $\overline{X} \setminus X$ is closed/equidim'l of codim c in $\overline{X}$). We then put $C^i_{\equiv}(X) := \dfrac{Z^i(\overline{X})}{Z^{i-c}(\overline{X} \setminus X) + Z^i_{\equiv}(\overline{X})}$, and remark that $C^{i-c}(\overline{X} \setminus X) \to C^i(\overline{X}) \to C^i(X) \to 0$ is exact.

Moreover, in the more general case where X & Y are quasi-projective: Prop. 2(c) still holds (for f^*), but for f_* to be defined we must assume that f is proper ("proper push-forward").