3. The Chow group

Given a smooth variety V/k and function $\varphi \in k(V)^*$, define

(1) $$\operatorname{div}(\varphi) := \sum_{\substack{W \subset V \\ \text{cd. } 1}} \operatorname{ord}_W(\varphi) \cdot W \in \operatorname{Div}(V)$$

(where writing $\varphi = \varphi_1/\varphi_2$, $\varphi_j \in \mathcal{O}_{W,V}$, $\operatorname{ord}_W(\varphi) = \operatorname{ord}_W(\varphi_1) - \operatorname{ord}_W(\varphi_2)$ and $\operatorname{ord}_W(\varphi_j) = \ell(\mathcal{O}_{W,V}/(\varphi_j))$). (1) is also valid for V singular.

Setting $D(V) \subset \operatorname{Div}(V)$ the subgroup generated by divisors of functions, we write

(2) $$CH^1(V) := \operatorname{Div}(V)/D(V).$$

Understanding the structure of this group is equivalent to solving the problem "when is a divisor the divisor of a rational function?". (There is also the analogy to ideal class groups in number theory.)

This definition was generalized to higher codimension through the work of Severi, followed by Chow and Samuel.

To define $\underset{\text{rat}}{\equiv}$ (rational equivalence), for any subvariety Y/k of X/k we shall write

$$\begin{tikzcd} \tilde{Y} \arrow[d] \arrow[dr, "\tilde{\imath}"] & \\ Y \arrow[r, hook, "\imath"'] & X \end{tikzcd}$$

for a resolution of singularities. Then

$$\left(Z^p(X) \supset \right) Z^p_{rat}(X) := \left\{ \sum_i^{(\text{finite})} \underset{\mathbb{Z}}{m_i} D_i \;\middle|\; \begin{array}{l} \exists\ Y_i \subset X \text{ irred cd. } p-1,\ \varphi_i \in k(Y_i)^* \\ \text{s.t. } D_i = \tilde{\imath}^i_* (\operatorname{div}(\varphi_i)) \; [= \imath^i_* (\operatorname{div}(\varphi_i))] \end{array} \right\}$$

(3)
$$= \left\{ Z \in Z^p(X) \;\middle|\; \begin{array}{l} \exists\ T \in Z^p(\mathbb{P}^1 \times X),\ a,b \in \mathbb{P}^1(k) \\ \text{s.t. } Z = T(b) - T(a) \end{array} \right\}.$$

(Note that we can always take $\{a,b\} = \{\infty, 0\}$. The "$\mathbb{P}^1$" is where the name "rational equivalence" comes from.) To see (3), note that if $Z = \tilde{\imath}_* \operatorname{div}(\varphi)$, then $T = (id_X \times \tilde{\imath})_* {}^t\Gamma_\varphi$ will suffice; whereas given T irred. with $\{a,b\} = \{\infty, 0\}$, $\pi_{\mathbb{P}^1}$ yields a function $\Phi \in k(T)^*$, and setting $\begin{cases} Y := (\pi_X)(T) \\ \varphi := \operatorname{Norm}_{k(T)/k(Y)} \Phi \end{cases}$ we have $\imath_* \operatorname{div}(\varphi) = \imath_* \pi_{Y*} (\operatorname{div} \Phi) = T(0) - T(\infty)$.

(RA_1) is clear for $\underset{rat}{\equiv}$ and I leave (RA_3) as an exercise. That leaves checking (RA_2), which is (Theorem 1:) <u>Chow's moving lemma:</u>
Let $\underset{\dim n}{X} \subseteq \mathbb{P}^N$, $Z \subset X$ irred dim p, $W \subset X$ irred. dim. q.
We must show that $\exists\ \tilde{Z} \underset{rat}{\equiv} Z$ s.t. $\tilde{Z} \cap W$ is proper.
If $X = \mathbb{P}^N$ we can do this by a (τ_ϵ) PGL_{n+1}-induced automorphism $\tau : \mathbb{P}^N \xrightarrow{\cong} \mathbb{P}^N$: indeed,

- there is a path — in fact, a sequence of lines — connecting any τ to $id_{\mathbb{P}^N}$, so that $\tilde{Z} := \tau(Z) \underset{rat}{\equiv} Z$; and
- we can choose τ_0 so that irred. components of $\tau_0(Z) \cap W$ contain a smooth point of both, then τ_1 so that the tangent spaces there meet properly.

For $X \subsetneq \mathbb{P}^N$ we apply the following inductively:

$$(4)\quad \begin{cases} \text{If } Z \cap W \text{ \underline{not} proper (i.e. } \dim > p+q-n\text{)}, \\ \text{then } \exists\, \mathfrak{Z}' \subset \mathbb{P}^N \text{ s.t. } \mathfrak{Z}' \cap X \text{ is proper and} \\ Z' := \mathfrak{Z}' \cdot X = Z + \sum m_j Z_j \text{ with } \begin{cases} \dim Z_j = d \\ Z_j \neq Z \end{cases} \text{ and } \dim Z_j \cap W < \dim Z \cap W. \end{cases}$$

[Applying τ as above to $\mathfrak{Z}'$ so that $\tau(\mathfrak{Z}') \cap W$ and $\tau(\mathfrak{Z}') \cap X$ are proper, $\left(Z = Z' - \sum m_j Z_j \underset{\text{rat}}{\equiv}\right) \underline{\tau(\mathfrak{Z}') \cdot X - \sum m_j Z_j}$ is "more proper" against W than was Z, and we now repeat on the Z_j.]

To prove (4), we choose projections (by choosing $p_k \notin X$ at each stage)

$$\left.\begin{array}{l} r_N : \mathbb{P}^N \setminus p_N \twoheadrightarrow \mathbb{P}^{N-1} \\ r_{N-1} : \mathbb{P}^{N-1} \setminus p_{N-1} \twoheadrightarrow \mathbb{P}^{N-2} \\ \quad \vdots \\ r_{n+1} : \mathbb{P}^{n+1} \setminus p_{n+1} \twoheadrightarrow \mathbb{P}^n \end{array}\right\} \rightsquigarrow \begin{array}{l} \Pi = r_{n+1} \circ \cdots \circ r_N : \mathbb{P}^N \setminus L_{\mathfrak{f}} \twoheadrightarrow \mathbb{P}^n \\ \pi = \Pi|_X : X \to \mathbb{P}^n \end{array}$$

equiv. choose $\tilde{p}_k \in \mathbb{P}^N$ w/ proj. linear span $L_{\mathfrak{f}} \cong \mathbb{P}^{N-n-1}$ not meeting X

same dim.

Pick $x_0 \in Z \subset X$, and x_j in each irred. component of $Z \cap Y$. We can choose the $\overset{(\sim)}{p}_k$ so that

(a) $\underline{\pi \text{ is finite}}$ $(\Rightarrow \text{flat})$ [each $r_k|_X$ has fibers $= \{\text{line}(\not\subset X) \cap X\}$]

(b) $\underline{\pi \text{ étale in Zariski open about each } x_j}$ [choose $\tilde{p}_k$'s so that $\mathrm{span}\langle L_{\mathfrak{f}}, x_j\rangle$ meets X transversely at x_j $(\forall j)$]

(c) $\underline{\pi|_Z : Z \xrightarrow{\cong} \pi(Z) \text{ over open nbhd. of each } \pi(x_j)}$ [choose $\tilde{p}_k$'s so that $\underset{\dim = N-n-1}{L_{\mathfrak{f}}}$ avoids $\underset{\dim \leq n}{\mathrm{cone}_{x_j}(Z)}$]

(d) $\underline{W \cap \pi^* \pi_* Z = (W \cap Z) \cup E, \; E \subset W \text{ closed } \& \; \mathrm{cd}_x E \geq \mathrm{cd}_x Z + \mathrm{cd}_x Y}$ $\underline{(\dim E \leq p+q-n)}$

[let $T = \{(w, z, \tilde{p}_N, \ldots, \tilde{p}_{n+1}) \mid w \in W \setminus Z,\ z \in Z,\ \tilde{p}_i \in \mathbb{P}^N,\ w \in \mathrm{span}\langle L_{\mathfrak{f}}, z\rangle\}$

$\downarrow \rho$

$B = (\mathbb{P}^N)^{N-n}$

compute $\dim T = p+q+(N-n)N - n$, so for general f
$\dim \rho^{-1}(f) = \dim T - \dim B = p+q-n$;
since $L_f \cap W = L_f \cap X = \emptyset$, $\dim \langle L_f, z\rangle \cap W =$ at most 0,
so $\rho^{-1}(f)$ is finite over W and image of dim $p+q-n$]

Now let $\mathfrak{Z}' = \Pi^{-1}\pi_* Z \subset \mathbb{P}^N$. Then

$$\text{(b)} \Rightarrow \mathfrak{Z}' \cap X \ (= \pi^*\pi_* Z) \text{ contains } Z \text{ with multiplicity } 1$$

$$\Rightarrow \mathfrak{Z}' \cdot X = Z + \sum m_j Z_j \quad \text{where } \dim \mathfrak{Z}_j = d \quad (\pi \underset{(a)}{\overset{\text{is}}{}} \text{finite!})$$

$$\Rightarrow Z_j \cap W \subset \pi^*\pi_* Z \cap W \underset{(d)}{\subset} (Z \cap W) \cup E$$

Write $Z_j \cap W$ as a sum of irreds. $\sum V_k$.

2 possibilities:

- $V_k \subset E \Rightarrow$ done ($\dim \leq p+q-n$)
- $V_k \subset Z \cap W \Rightarrow V_k \not\ni x_i$'s by (b) & (c) $\left[\begin{array}{l}\Rightarrow \text{for } f \in \text{Zar. open}, \\ Z \to \pi^{-1}\pi Z \text{ is étale} \\ \text{at } x_i\end{array}\right]$

  $\Rightarrow \dim V_k < \dim Z \cap W$.

$\square$

<u>Definition 1</u>: $CH^p(X) := Z^p(X) / Z^p_{rat}(X)$. $\left[\text{Also: } CH^p_{\equiv}(X) := Z^p_{\equiv}(X) / Z^p_{rat}(X)\right]$

We now turn to an example of the computation of a Chow group.

Note that if $\underset{\dim d}{X} \subset \mathbb{P}^{d+1}$ has degree D, then

$$K_X \underset{\text{adjunction}}{\cong} K_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(X)\big|_X \cong \mathcal{O}_{\mathbb{P}}(-d-2) \otimes \mathcal{O}_{\mathbb{P}}(D)\big|_X = \mathcal{O}_X\big((D-d-2)\underset{\substack{\uparrow \\ \text{hyperplane} \\ \text{section}}}{H}\big)$$

$$\Rightarrow h^{d,0}(X) = \dim \Gamma(X, K_X) = S^{D-d-2}_{d+2}$$

$$= \begin{cases} 0 & , D \leq d+1 \\ \binom{D-1}{d+1}, & D > d+1 \end{cases}$$

[homogeneous polynomials of degree $D-d-2$ in $d+2$ variables]

<u>Theorem 2 (Roitman)</u>: For a hypersurface $X \subset \mathbb{P}^{d+1}$ of degree $D \leq d+1$,

$$(CH_0(X) =) CH^d(X) \xrightarrow[\deg]{\cong} \mathbb{Z}.$$

$$\sum m_i p_i \longmapsto \sum m_i$$

Proof: Assume $1 < D \leq d$.*

Step 1 Consider the local affine equation of X about $p \in X(k)$:

$$0 = f_1(\underline{x}) + \cdots + f_D(\underline{x})$$

To say that $\lambda_{\underline{a}} : t \mapsto (a_0 t, \ldots, a_d t)$ belongs to X means

$$0 = t f_1(\underline{a}) + \cdots + t^D f_D(\underline{a}) \qquad (\forall t),$$

i.e. $[\underline{a}]$ belongs to the complete intersection $V(f_1, \ldots, f_D) \subset \mathbb{P}^d$.

Since $D \leq d$, and $k = \bar{k}$, $V \neq \emptyset$.

Step 2 Let $p, q \in X(k)$, $L_p, L_q \subset X$ thru p, q resp.

$H^2(\mathbb{P}^{d+1}) \cong \mathbb{Z}$, $CH_1^{hom}(\mathbb{P}^{d+1}) = \{0\} \Rightarrow L_p \underset{rat}{\equiv} L_q$ on $\mathbb{P}^{d+1}$.

Now L_p and L_q of course don't properly intersect X, but we can use Fulton's refined intersection product to have (on X)

$L_p \cdot X \underset{rat}{\equiv} L_q \cdot X$, where $\left\{\begin{matrix} L_p \cdot X \\ L_q \cdot X \end{matrix}\right.$ is a 0-cycle of degree D supported on $\left\{\begin{matrix} L_p \\ L_q \end{matrix}\right.$. Since every point on L_p (resp. L_q) is $\underset{rat}{\equiv} p$ (resp. $\underset{rat}{\equiv} q$), we get $D \cdot p \underset{rat}{\equiv} D \cdot q$ on X.

Step 3 Since $CH_0^{hom}(X)$ is generated by differences of points $p - q$, we conclude that $D \cdot CH_0^{hom}(X) = 0$. Since $CH_0^{hom}(X)$ is divisible (see below), it follows that $CH_0^{hom}(X) = 0$. Hence for any $P, Q \in X(k)$, $P \underset{rat}{\equiv} Q$. $\square$

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* $D = d+1$ is an exercise, using

$$\{\lambda_{\underline{a}} \subset X \ \text{or}\ \lambda_{\underline{a}} \cap X = D \cdot p\} \iff [\underline{a}] \in V(f_1, \ldots, f_{D-1})$$

Remark 1:

The divisibility assertion is checked as follows: given $P, Q \in X(k)$, take a curve C thru P, Q. We then have

$$\begin{array}{ccc} J(C) & \longrightarrow & CH_0^{hom}(X) \\ [P-Q] & \longmapsto & [P-Q] \end{array},$$

which shows that all the generators of CH_0^{hom} lie in the image of a torus (over $k=\bar{k}$). Since tori are divisible, so is CH_0^{hom}. //

> **Note:** divisible groups can have torsion, but they cannot BE torsion.

Remark 2: We note the exact sequence (for $Y \subset X$ closed)

$$(5) \qquad CH_q(Y) \to CH_q(X) \to CH_q(X \setminus Y) \to 0$$

which follows from the generalities of §I.A.2.

Remark 3: $\underset{rat}{\equiv}$ is the "finest" adequate equivalence relation. Here are the subgroups corr. to a few others:*

$$Z^i_{rat} \subsetneq Z^i_{alg} \subsetneq Z^i_{\otimes} \subseteq Z^i_{hom} \subseteq Z^i_{num} \subsetneq Z^i$$

- Z^i_{alg} is the "same" definition as (3) but with the $\mathbb{P}^1$ replaced by an arbitrary sequence of algebraic curves
- Z^i_{hom} is homological equivalence (regard alg. cycles as topological ones: $Z_1 \underset{hom}{\equiv} Z_2$ if $\exists$ C^∞ chain Γ with $\partial\Gamma = Z_1 - Z_2$).
- you can look up the rest in Murre's article
- Z^i_{hom} / Z^i_{alg} is called the Griffiths group $Griff^i(X)$. Trivial for divisors + 0-cycles.

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* "$\subsetneq$" means there exist X for which it isn't $=$.

Exercise: show (in detail) how to use 3 $\mathbb{P}^2$'s to prove Theorem 2 without the refined intersection product. (Take $D > 1$ and $\leq d$, and aim at $(D-1)(p-q) \underset{rat}{\equiv} 0$.)

Exercise: prove theorem 2 in the case $D = d+1$, using the hint on page 5.

As we shall see, Theorem 2 is a first instance of the way in which the Hodge theory of X (in this case, vanishing of $h^{d,0}(X)$) "controls" the cycle groups on X. Indeed, as we shall see, if $h^{d,0}(X) \neq 0$ then $CH_0^{hom}(X)$ becomes "huge", while any sort of converse (see §I.C) is still conjectural — but with good evidence like Theorem 2.