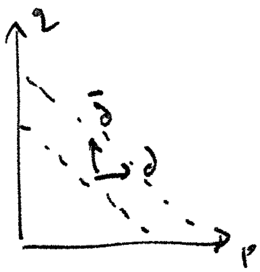
B. Cycle class map and the Hodge Conjecture

1. Cycle class and Lefschetz (1,1)

Recall the decomposition of $\mathbb{C}$-valued C^∞ k-forms on $X^{an}_{\mathbb{C}}$: (will sometimes be tacit)

(1) $$A^k(X) = \bigoplus_{p+q=k} \underbrace{A^{p,q}(X)}_{\{\sum f_{IJ}\, dz_I \wedge d\bar{z}_J\}} \;, \qquad \overline{A^{p,q}} = A^{q,p}$$



$$d = \partial + \bar\partial : A^k \to A^{k+1}, \qquad H^k(X,\mathbb{C}) := H^k\{A^\bullet(X), d\}.$$

$$H^{p+q}(X,\mathbb{C}) \supset H^{p,q}(X,\mathbb{C}) := \frac{\ker(d) \subset A^{p,q}(X)}{dA^{p+q-1}(X) \cap A^{p,q}(X)}$$

$\cong$ Dolbeault $H^q(X, \Omega^p_X)$ (alternatively, $= \partial\bar\partial A^{p-1,q-1}(X)$)

Assume X proper (or at least compact Kähler!). Each class in H^{p+q} has a unique Δ-harmonic representative $\alpha = \sum \alpha^{(p,q)}$, and $\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial} \implies \Delta\alpha^{(p,q)} = 0$
$\implies \mathcal{H}^k = \bigoplus \mathcal{H}^{p,q}$ (harmonic forms) $\implies$ (1) descends to cohomology

(2) $$H^k(X,\mathbb{C}) = \bigoplus H^{p,q}(X) \qquad \left(\overset{\text{dims}}{\implies} h^k(X) = \textstyle\sum h^{p,q}(X)\right).$$

This already says some interesting things about the topology of alg. varieties: $H^{p,q} = \overline{H^{q,p}} \implies h^{p,q} = h^{q,p} \implies$

(3) for X proper & k odd, $h^k(X)$ is even.

Recall the Hodge filtration

(4) $$F^a A^k(X) = \bigoplus_{p \geq a} A^{p,k-p}(X), \qquad F^a H^k(X,\mathbb{C}) := \bigoplus_{p \geq a} H^{p,k-p}(X).$$

We also get a refinement of Poincaré duality; recall this says that for X proper/smooth* (dim n)

$$
(5)\quad \left\{ \begin{array}{lll}
H_k(X) \times H_{2n-k}(X) & \longrightarrow & \mathbb{C} \\
(\gamma\ ,\ \tau) & \longmapsto & \#(\gamma \cap \tau) \quad \text{(if meet transversely)} \\
H_k(X) \times H^k(X) & \longrightarrow & \mathbb{C} \\
(\gamma\ ,\ \omega) & \longmapsto & \int_\gamma \omega \\
H^k(X) \times H^{2n-k}(X) & \longrightarrow & \mathbb{C} \\
(\eta\ ,\ \omega) & \longmapsto & \int_X \eta \wedge \omega
\end{array} \right.
\qquad \left(\underline{\text{also}}:\ H^r(X,\mathbb{Z}) \cong H_{2n-r}(X,\mathbb{Z})\right)
$$

are perfect pairings (i.e. nondegenerate). The refinement is that

$$
(6)\qquad H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C}
$$

is perfect as well ($\forall p,q$).

Now take $\gamma \in Z_r^{top}(X^{an}_{\mathbb{C}})$ a topological cycle ($\underline{\text{real}}$ dim. r), $[\gamma] \in H_r(X,\mathbb{Z})$ its class. Integration over γ gives a functional on r-forms hence on $H^r(X,\mathbb{C})$. In particular, if $V \overset{i}{\subset} X$ is a subvariety of (complex) cod. k, then

$$
(7)\qquad
\begin{tikzcd}
{[V]} \arrow[r, phantom, "\in"] \arrow[ddrrr, mapsto, bend right] & H_{2n-2k}(X,\mathbb{Z}) \arrow[rr, "\text{P.D.}"] \arrow[d] & & H^{2k}(X,\mathbb{Z}) \arrow[d, "\text{not nec. injective: torsion may be killed}"] \\
& H_{2n-2k}(X,\mathbb{C}) \arrow[r, "\cong", "\int"'] & H^{2n-2k}(X,\mathbb{C})^\vee \arrow[r, "\cong"] & H^{2k}(X,\mathbb{C}) \\
& & & \int_V
\end{tikzcd}
$$

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* I take coeffs. in $\mathbb{C}$ here — they could also be in $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{Q}_\ell$ ...

Now consider the decomposition of $\int_V$ under

$$\begin{array}{ccc} H^{2k}(X,\mathbb{C}) & = & \bigoplus_p H^{p,2k-p} \\ \uparrow\cong & & \uparrow\cong \\ H^{2n-2k}(X,\mathbb{C})^\vee & = & \bigoplus_p \left(H^{n-p,n-2k+p}\right)^\vee \end{array}$$

The point is that forms of type $(n-p, n-2k+p) \neq (n-k, n-k)$ have too many dz's & $d\bar{z}$'s, and pull back to zero on V. So

$$\int_V \in \operatorname{image}\left\{H^{k,k}(X) \hookrightarrow H^{2k}(X,\mathbb{C})\right\},$$

and (7) extends by linearity to give a map

$$(8) \qquad \widetilde{cl}^k_X : Z^k(X) \longrightarrow H^{2k}(X,\mathbb{Z}) \cap \overset{\text{in } H^{2k}(X,\mathbb{C})}{\downarrow} H^{k,k}(X,\mathbb{C}) =: Hg^k(X)$$

Actually, the definition of $Hg^k(X)$ is really (to deal w/ torsion)

$$Hg^k(X) := \ker\left\{H^{2k}(X,\mathbb{Z}) \oplus \underset{\text{or } F^k H^{2k}(X,\mathbb{C})}{H^{k,k}(X)} \to H^{2k}(X,\mathbb{C})\right\}$$

and " $\widetilde{cl}^k_X(V) := \big(\underset{\text{as top. cycle}}{[V]_{top}}, \underset{\text{as current of integration}}{\delta_V}\big)$ ".

Now if $Z \equiv_{rat} 0$, then $\exists\ T \in Z^k(\mathbb{P}^1 \times X)$ s.t. $Z = T(0) - T(\infty)$

$\Rightarrow \partial T(\overrightarrow{\infty.0}) = T(\partial(\overrightarrow{\infty.0})) = T(0-\infty) = T(0) - T(\infty) = Z$

$\Rightarrow Z \underset{hom}{\equiv} 0$. So $\widetilde{cl}^k_X$ factors thru $\underset{rat}{\equiv}$ and we have

<u>Definition 1 (cycle class map)</u>: Define

$$cl^k_X : CH^k(X) \to Hg^k(X)$$

to be the map induced by $\widetilde{cl}$.

The group $Hg^k(X)$ indicates "symmetries of periods", which is an analytic/topological phenomenon. Evidence suggests that this predicts something about the algebraic structure of X:

Hodge Conjecture: cl^k_X is rationally surjective.*

Theorem 1 (Lefschetz (1,1)): $cl^1_X : Div(X) \to Hg^1(X)$ is surjective.

Proof (Kodaira/Spencer): Uses hard $\exists$ theorem: every holomorphic line bundle is of the form L_D, $D \in Div(X)$ — equiv., L has a mero. section. (*)

Write down the exact sequence

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{e^{2\pi i(\cdot)}} \mathcal{O}_X^* \to 0$$

of sheaves on X, and look at the long exact sequence portion

$$\begin{tikzcd}
 & \to H^1(X,\mathcal{O}_X^*) \arrow[r, "{[\cdot]}"] & \ker\left\{H^2(X,\mathbb{Z}) \to \overbrace{H^2(X,\mathcal{O})}^{H^{0,2}}\right\} \arrow[r] & 0 \\
L_D & \left\{\substack{\text{holo. line}\\ \text{bundles}}\right\} \arrow[u, equal] & Hg^1(X) \arrow[u, "\cong"'] & \\
D \in \arrow[u, mapsto] & CH^1(X) \arrow[u, two heads, "(*)"'] \arrow[ur, "cl^1_X"'] & &
\end{tikzcd}$$

$cl^1_X \Leftarrow [L_D] = cl(D)$

□

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* Fails integrally in general — result of Atiyah & Hirzebruch using their spectral sequence for topological K-theory. Deligne suggests a refinement of HC taking their method of proof into account.