2. Mumford – Tate groups of Hodge structures

Since they will be used to explain proofs of some cases of the Hodge Conjecture for abelian varieties, and later in our discussion of normal functions, we now introduce Mumford–Tate groups – a sort of symmetry or Galois group (in the Tannakian sense) associated to Hodge structures.

First we briefly discuss algebraic groups. An <u>algebraic group</u> G over a field k (of characteristic zero) is a smooth algebraic variety G together with morphisms

$$(1) \quad \begin{cases} \cdot : G \times G \to G \\ (\cdot)^{-1} : G \to G \end{cases} \quad \text{defined} / k \qquad \left(\begin{array}{c} \text{morphisms} \\ \mathrm{Spec}\, L \to G \end{array} \right)$$

and element $1_G \in G(k)$, subject to rules making $G(L)$ ($=$ L-rational pts. of G) into a group for any L/k. $G(\mathbb{R})$ and $G(\mathbb{C})$ admit the structure of real resp. complex Lie groups.

Exercise: Write out these rules as commutative diagrams.

Example 1 ($GL_1 \cong$) $\mathbb{G}_m := \{XY = 1\} \subset \mathbb{A}^2$, $\quad \mathbb{G}_m(k) = (k^*, \cdot)$

$\mathbb{G}_a := \mathbb{A}^1$, $\quad \mathbb{G}_a(k) = (k, +)$ //

- G is <u>connected</u> $\overset{\text{def.}}{\iff}$ $G_{\bar{k}}$ irreducible [as alg. variety]

$$\left(G \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} \bar{k} \right)$$

- G is <u>simple</u> $\overset{\text{def}}{\iff}$ G is nonabelian w/ no normal connected subgroups

<u>Example 2</u>/ $k = \mathbb{C}$:

		Cartan type
classical	SL_2	A
	SO_n	B, D
	Sp_n	C
exceptional	E_6, E_7, E_8	
	F_4	
	G_2	

$k = \mathbb{R}$: $\exists$ distinct real forms of these groups which are isomorphic/$\mathbb{C}$ but not isomorphic over $\mathbb{R}$, e.g.

$SU(p,q)$ for $SL(p+q)$

$SO(p,q)$ for $SO(p+q)$,

but no simple groups /$\mathbb{R}$ which become non-simple /$\mathbb{C}$

$k = \mathbb{Q}$: $\mathbb{Q}$-simple $\not\Rightarrow$ $\mathbb{R}$-simple,

so it's not just an issue of $\mathbb{Q}$-forms. //

<u>Remark 1</u>: In general, for an extension E/F, the F-forms of a group G/E are parametrized by $H^1(\mathrm{Gal}(E/F), \mathrm{Aut}(G))$.

- G (algebraic) <u>torus</u> $\overset{\text{def.}}{\iff}$ $G_{\bar{k}} \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$

<u>Example 3</u>/ $k = \mathbb{Q}$: for any number field E, $G := \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$ is an irreducible torus of dim $[E:\mathbb{Q}]$ with the property that $G(\mathbb{Q}) \cong E^\times$, $G(k) \cong E^\times \otimes_{\mathbb{Q}} k$ $\left(= (k^\times)^{[E:\mathbb{Q}]} \text{ if } k \supset E \right)$. //

Now let $k \subset \mathbb{C}$: inside $GL(2)$, we have tori

$$\mathbb{U} \subset \mathbb{S} \supset \mathbb{G}_m$$

with

$$(2)\quad \begin{cases} \mathbb{U}(k) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, \begin{array}{l} a^2+b^2=1 \\ a,b \in k \end{array} \right\} \\ \quad \downarrow \\ \mathbb{S}(k) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, \begin{array}{l} a^2+b^2 \neq 0 \\ a,b \in k \end{array} \right\} \\ \quad \uparrow \\ \mathbb{G}_m(k) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \,\middle|\, \alpha \in k^* \right\} \end{cases}$$

– in particular, complex pts

$$(3)\quad \begin{array}{ccccc} \mathbb{C}^* & \hookrightarrow & \mathbb{C}^* \times \mathbb{C}^* & \hookleftarrow & \mathbb{C}^* \\ z & \longmapsto & (z, z^{-1}) & & \\ & & (\alpha, \alpha) & \longleftarrow & \alpha \end{array}$$

(take eigenvalues of matrices)

and real pts.

$$(4)\quad \begin{array}{ccccc} S^1 & \hookrightarrow & \mathbb{C}^* & \hookleftarrow & \mathbb{R}^* \end{array}.$$

(or "$U(1)$")
$\shortparallel$
unit circle in $\mathbb{C}^*$

There is a map

$$(5)\quad \begin{array}{ccc} \mathbb{U} & \xrightarrow{\cong} & \mathbb{G}_m \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & \longmapsto & \begin{pmatrix} a+bi & 0 \\ 0 & a+bi \end{pmatrix} \end{array} \qquad \text{defined } /\mathbb{Q}(i) \text{ (or } \mathbb{C})$$

but <u>NOT</u> $\mathbb{Q}$ (or $\mathbb{R}$). This reflects the fact that $\mathbb{U}$ and $\mathbb{G}_m$ are the 2 $\mathbb{R}$-forms of $\mathbb{G}_{m,\mathbb{C}}$, parametrized by

$$H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}(\mathbb{G}_m)) \cong \mathbb{Z}/2\mathbb{Z}.$$

- G is <u>semisimple</u> $\overset{\text{def.}}{\iff}$ G is an almost-direct product of simple groups $\left(G_1 \times \cdots \times G_n \underset{\text{isogeny}}{\twoheadrightarrow} G \right)$

- G is <u>reductive</u> $\overset{\text{def.}}{\iff}$ G is an almost-direct product of simple groups and tori

  $\underset{\text{FACT}}{\iff}$ finite-dimensional linear representations of G are completely reducible.

<u>Example 4</u>/ One finite dimensional representation is the <u>adjoint map</u>

(6) $$\mathrm{Ad}: G \longrightarrow GL(\mathfrak{g}) \quad , \quad \mathfrak{g} = \mathrm{Lie}\, G = T_e G \quad (\text{vector space}/k)$$

$$g \longmapsto \underbrace{\{X \mapsto gXg^{-1}\}}_{\text{differential of } \Psi_g \in \mathrm{Aut}(G)}$$

image $=: G^{ad}$. //

- Assume G semisimple: then G <u>adjoint</u> $\overset{\text{def.}}{\iff}$ $G = G^{ad}$ (Ad injective)

  – e.g. PSL_2.

- With the same assumption: G <u>simply connected</u> $\overset{\text{def.}}{\iff}$ any isogeny $G' \twoheadrightarrow G$ with G' connected is an $\cong$.

  – e.g. SL_2.

There is always a finite tower of isogenies

$$\underset{\text{largest poss. } Z(G)}{G^{sc}} \twoheadrightarrow \cdots \cdots \twoheadrightarrow \underset{Z(G)=1}{G^{ad}} .$$

Let $V = \mathbb{Q}$ - vector space, $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$.

<u>Definition 1</u>: A Hodge structure of weight n on V is (equiv.)

(i) a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ where $\overline{V^{p,q}} = V^{q,p}$

(ii) a decreasing filtration $\cdots \supseteq F^p V_{\mathbb{C}} \supseteq F^{p+1} V_{\mathbb{C}} \supseteq \cdots$
satisfying $V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{n-p+1} V_{\mathbb{C}}}$

(iii) a homomorphism of <u>real</u> algebraic groups $\tilde{\varphi} : \mathbb{S} \to GL(V)$
s.t. $\tilde{\varphi}\left(\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\right) = r^n$. [or: $\varphi : \mathbb{U} \to SL(V)$ s.t. $\varphi(-1) = (-1)^n$]

<u>Exercise</u>. Work out the details of the equivalences.

[(i) $\Leftrightarrow$ (iii) : the idea is that (regarding $z \in S^1 = \mathbb{U}(\mathbb{R})$)
$\varphi(z)$ = mult. by z^{p-q} on $V^{p,q}$]

We write $h^{p,q} = \dim_{\mathbb{C}} V^{p,q}$, $h = \sum_{p+q=n} h^{p,q} = \dim_{\mathbb{Q}} V$, and <u>if</u> <u>$n$ even</u>, $h^{\text{even}} := \sum_{\substack{p+q=n \\ p \text{ even}}} h^{p,q}$, $h^{\text{odd}} := \sum_{\substack{p+q=n \\ p \text{ odd}}} h^{p,q}$.

Now given
- (V, φ) HS of weight n
- $Q : V \otimes V \to \mathbb{Q}$ nondegenerate bilinear form (symm if n even; antisymm if n odd)

<u>Definition 2</u>: Q is a <u>polarization</u> of (V, φ) if the Hodge-Riemann bilinear relations hold:

(HR_1) $\quad Q(V^{p,q}, V^{p',q'}) = 0$ unless $(q,p) = (p',q')$

(HR_2) $\quad \sqrt{-1}^{\,p-q}\, Q(v, \bar{v}) > 0$ for any $v \in V^{p,q} \setminus \{0\}$.

We say (V, φ, Q) is a PHS (polarized Hodge structure).

Remark 2: A sub-Hodge structure is $W \subset V$ (def'd $/\mathbb{Q}$) s.t.

$$W_{\mathbb{C}} = \bigoplus_{p+q=n} W_{\mathbb{C}} \cap V^{p,q},$$

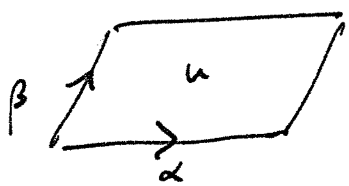
equiv. $\varphi(S^1) W_{\mathbb{C}} \subset W_{\mathbb{C}}$. By taking $\perp$ complements w.r.t. Q, the category of PHS is seen to be semisimple.

Proposition 1: (V, φ, Q) PHS $\Rightarrow$ φ factors thru $\mathrm{Aut}(V, Q) =: G$

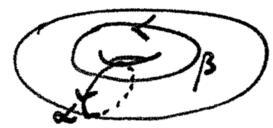
(Remark that this [$\mathbb{Q}$-algebraic] group has $\mathbb{R}$-points $(G(\mathbb{R}))$ $\cong \begin{cases} SO(h^{\text{even}}, h^{\text{odd}}), & n \text{ even} \\ Sp_h, & n \text{ odd} \end{cases}$.)

Proof: WTS $Q(\varphi(z)v, \varphi(z)v') = Q(v, v')$ $\forall\, v, v' \in V_{\mathbb{C}}$.
This follows from (HR_1) by expanding v, v' by type. $\square$

Example 5/ (Elliptic curves)



$\mathbb{C}/\mathbb{Z}\langle 1, \tau\rangle \xrightarrow{(P, P')} E_\tau$



(transcendental map!)

$V := H^1(E_\tau, \mathbb{Q}) = \mathbb{Q}\langle \alpha^*, \beta^*\rangle$

$V_{\mathbb{C}} = \Omega^1(E_\tau) \oplus \overline{\Omega^1(E_\tau)} = \mathbb{C}\langle \underset{du}{\underset{\shortparallel}{\omega}}\rangle \oplus \mathbb{C}\langle \bar{\omega}\rangle$

$[\varphi(z)]_{\{\omega, \bar{\omega}\}} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$, $\quad [Q]_{\{\alpha^*, \beta^*\}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

then (V, φ, Q) is a PHS $\iff$ $\tau \in \mathfrak{h}$. //

Remark 3: All compact complex 1-tori are algebraic (ell. curves).

In general,

weight 1 rank $2g$ HS $\leftrightarrow$ compact cx. g-tori
weight 1 rank $2g$ PHS $\leftrightarrow$ abelian g-fold

parameters
g^2
$\binom{g+1}{2} = \dim_{\mathbb{C}} \mathfrak{h}_g$

Let (V, φ, Q) be a PHS. Put

tensor spaces $\quad T^{a,b} V := V^{\otimes a} \otimes \check{V}^{\otimes b} \qquad$ ($\mathbb{Q}$-vector spaces)

Hodge tensors (?) $\quad Hg^{a,b} V := \begin{cases} T^{a,b}V \cap (T^{a,b} V_{\mathbb{C}})^{m,m}, & (a-b)n = 2m \\ 0 & , (a-b)n \text{ odd}. \end{cases}$

"MTG"

<u>Definition 3</u>: The <u>Mumford-Tate group</u> M_φ of (V, φ) is the $\mathbb{Q}$-algebraic closure of the image of φ, i.e. the smallest ($\mathbb{Q}$-) algebraic subgroup of $\mathrm{Aut}(V, Q)$ whose real pts. contain $\varphi(S^1)$ [cx. pts. contain $\varphi(\mathbb{C}^*)$].

<u>Proposition 2</u>: (a) M_φ is the (largest) $\mathbb{Q}$-algebraic subgroup of $\mathrm{Aut}(V, Q)$ fixing all $Hg^{a,b} V$ pointwise.

(b) M_φ is connected and reductive.

<u>Proof</u>: (a) uses <u>Chevalley's theorem</u>: Any closed $\mathbb{Q}$-alg. sgp. of $GL(V)$ is the stabilizer of a line L in a finite direct sum $\bigoplus_{i=1}^{m} T^{k_i, k_i}$.

Suppose $t \in Hg^{k,l} V$. Then

t rational $\Rightarrow$ fixing t defines $\mathbb{Q}$-alg. sgp. $\mathrm{Fix}(t) \subseteq GL(V)$.

t (p,p) $\Rightarrow$ $\varphi(S^1)$ fixes t $\underset{\text{(Def. 3)}}{\Rightarrow}$ $M_\varphi \subset \mathrm{Fix}(t)$. So

(8) $$M_\varphi \subseteq \mathrm{Fix}(Hg^{\cdot,\cdot} V)$$

Now by Chevalley,

(9) $$M_\varphi = \mathrm{Stab}(t_1, \dots, t_m) \subseteq \cap\, \mathrm{Stab}(t_i)$$

$\Rightarrow$ $\varphi(S^1)$ stabilizes each t_i

$\Rightarrow$ each t_i of pure Hodge type, hence (since rational) of type (p_i, p_i)

$\Rightarrow$ $t_i \in Hg^{k_i,k_i} V$ $(\forall i)$

So
$$\mathrm{Fix}(Hg^{\cdot,\cdot} V) \subseteq \cap\, \mathrm{Fix}(t_i) \underset{(9)}{\subseteq} M_\varphi \underset{(8)}{\subseteq} \mathrm{Fix}(Hg^{\cdot,\cdot} V),$$
done.

(b) [sketch] $\underset{/\mathbb{Q}}{W \subset V}$ is a sub HS $\iff \varphi(S^1) W_{\mathbb{C}} \subset W_{\mathbb{C}}$

$\underset{\text{(Def. 3)}}{\iff} M_\varphi W \subset W$.

Since all linear representations of M_φ are contained in $T^{\cdot,\cdot} V$, it's enough to know each $W \subset T^{k,l} V$ has a complementary sub HS. This is given (as $W^\perp$) by the polarization Q.

□

Example 5 (cont'd.)

To compute the MTG of $V = H^1(E)$, note that the only nontrivial subgroups of $\mathrm{Aut}(V, Q) = SL_2$ are (SL_2 itself and) 1-tori, and they are cut out by elements of $T^{1,1}$. Now Q gives isomorphisms

$$T^{2,0} \xleftrightarrow{\cong} T^{1,1} \xleftrightarrow{\cong} T^{0,2}$$

$$\frac{\omega\wedge\bar\omega}{\bar\tau-\tau} = \alpha\wedge\beta \quad\longleftrightarrow\quad \mathrm{id}_V \quad\longleftrightarrow\quad Q$$

$\leftarrow$ Hodge tensors (these cut nothing out of SL_2)

What other Hodge tensors are there (besides constant multiples of the ones shown)? In the $\{\omega,\bar\omega\}$ basis we need $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ commuting with $\begin{pmatrix} z & \\ & z^{-1}\end{pmatrix}$ $\forall z \in S^1$, in order to get a new element of $T^{1,1}$ of Hodge type $(0,0)$. That is, we must have $b = c = 0$. In order for this tensor to be defined $/\mathbb{R}$, we must have $d = \bar a$.

Now change basis to $\{\alpha,\beta\}$:

$$(10)\quad \frac{1}{\bar\tau-\tau}\begin{pmatrix} -\tau & -\bar\tau \\ 1 & 1\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & \bar a\end{pmatrix}\begin{pmatrix} 1 & \bar\tau \\ -1 & -\tau\end{pmatrix} = \frac{1}{\bar\tau-\tau}\begin{pmatrix} \bar\tau\bar a - \tau a & |\tau|^2(\bar a - a) \\ a-\bar a & a\bar\tau - \bar a\tau\end{pmatrix}$$

must belong to $M_2(\mathbb{Q})$, which implies $\mathrm{Im}(a) \in \mathbb{Q}\,\mathrm{Im}(\tau)$, in order to have a ($\mathbb{Q}$-) Hodge tensor; and for it to be different from $\mathbb{Q}\cdot \mathrm{id}_V$ we need $\mathrm{Im}(a) \neq 0$. Writing $\tau = C + Di$, $a = A + Bi$ we may assume $B = D$, whereupon $(10) \in M_2(\mathbb{Q})$

$$\Rightarrow \begin{cases} C^2 + D^2 \in \mathbb{Q} \\ C - A \in \mathbb{Q} \\ C + A \in \mathbb{Q}\end{cases} \Rightarrow \quad \begin{matrix} C, A \in \mathbb{Q} \\ D^2 \in \mathbb{Q}\end{matrix}.$$

Since $\tau \in \mathfrak{h}$, we must have $D \in \mathbb{R}$ (not $i\mathbb{R}$), and so $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$.

We have shown

$$[\mathbb{Q}(\tau):\mathbb{Q}] > 2 \implies M_\varphi = SL_2 \;;$$

Conversely, if $[\mathbb{Q}(\tau):\mathbb{Q}] = 2$, then taking $a = \tau$ gives a Hodge tensor (plug into (10)), corresponding to multiplication by τ on (the complex tangent space of) E_τ. We say E_τ is of CM (complex multiplication) type.

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In preparation for the next section we conclude with a few more generalities, related to the ring of endomorphisms

$$\mathcal{E}_\varphi := \mathrm{End}(V,\varphi) = \mathrm{End}(V)^{M_\varphi} = Hg^{1,1}V.$$

WARNING: $\mathcal{E}_\varphi$ is not a group, is not contained in M_φ, and does not preserve Q (consider multiplication by N on V !).

Definition 4: (a) A CM field is a totally imaginary extension of a totally real number field.

(b) A CM Hodge structure is a (V,φ) such that M_φ is abelian ($\Rightarrow$ algebraic torus).

No relationship between (a) and (b) seems obvious, but using the fact that $\mathcal{E}_\varphi$ is a division algebra for a simple/irreducible

(synonyms)

(V, φ), together with the Rosati involution $\mathcal{E}_\varphi \xrightarrow{\dagger} \mathcal{E}_\varphi^{op}$ defined by $Q(\varepsilon v, w) = Q(v, \varepsilon^\dagger w)$, one deduces the

Proposition 3*: If (V, φ, Q) is irreducible and CM, then $\mathcal{E}_\varphi$ is isomorphic to a CM field.

We say (V, φ) has K-multiplication (by a number field K) if $\exists$ embedding $K \hookrightarrow \mathcal{E}_\varphi$.

Proposition 4**: We have $[K:\mathbb{Q}] \leq \dim_{\mathbb{Q}} V$. If equality holds, then K and (V, φ) are CM, and $K \cong \mathcal{E}_\varphi$. [Note: here the HS must be polarizable.]

More generally, suppose a PHS has decomposition

$$(V, \varphi) = (V_1^{\oplus m_1} \oplus \cdots \oplus V_\ell^{\oplus m_\ell}, \varphi_1^{\oplus m_1} \oplus \cdots \oplus \varphi_\ell^{\oplus m_\ell})$$

into simple factors. Then $\mathcal{E}_\varphi \cong M_{m_1}(\mathcal{E}_{\varphi_1}) \times \cdots \times M_{m_\ell}(E_{\varphi_\ell})$, where $\mathcal{E}_{\varphi_i}$ are division algebras of the following types:

Albert classification of simple $\mathbb{Q}$-algebras with "positive involution" [this is the "$\dagger$" above]

- (I) totally real number field
- (II), (III) quaternion algebra over totally real number field, which $\left\{\begin{array}{l}\text{splits } (\cong M_2(\mathbb{R})) \\ \text{is inert } (\cong \mathbb{H})\end{array}\right\}$ under all real embeddings of the # field
- (IV) CM field or division algebra over one

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*Cf. [GGK], (V.2).   ** cf. [GGK], (V.3)

Finally, let (V, φ, Q) be a PHS and $G = \mathrm{Aut}(V, Q)$.

We have $M_\varphi = \mathrm{Fix}(Hg^{\bullet}V) \leq G$. Define the Lefschetz group by

$$(11) \quad L_\varphi := \mathrm{Fix}(Hg^{1,1}V) \left(= G^{\mathcal{E}_\varphi} = Sp_{\mathcal{E}_\varphi}(V, Q) \right) \leq G.$$

Clearly $L_\varphi \geq M_\varphi$ — i.e. it gives an "upper bound" for the Mumford–Tate group — and (via Q) $L_\varphi = \mathrm{Fix}(Hg^{2,0}V) = \mathrm{Fix}(Hg^{0,2}V)$. So L_φ is the group cut out of G by the property of fixing all Hodge tensors of degree 2.

Definition 5: (V, φ, Q) is nondegenerate $\iff M_\varphi = L_\varphi$.

Example 6/ If E_1, E_2, E_3 are pairwise nonisogenous CM elliptic curves, $H^1(E_1) \otimes H^1(E_2) \otimes H^1(E_3)$ is a degenerate CM HS. //

Mumford–Tate groups were originally introduced by Mumford in a 1966 paper to give a Hodge-theoretic characterization of families of abelian varieties studied by Shimura, and some that Shimura had missed (characterized by Hodge tensors of degree > 2).

The idea (in some sense) goes back further, to Picard, who studied the moduli of curves of the form

$$C := \overline{\{ y^3 = x(x-1)(x-\alpha)(x-\beta) \}} \subset \mathbb{P}^2.$$

The cubic automorphism induced by $y \mapsto \overset{\text{"}\zeta_3\text{"}}{e^{2\pi i/3}}\, y$ leads to an embedding $\mathbb{Q}(\zeta_3) \hookrightarrow \mathcal{E}_\varphi$ (where $(V,\varphi) \leftrightarrow H^1(C)$), with a corresponding eigen-decomposition $V_{\mathbb{Q}(\zeta_3)} = V_+ \oplus V_-$ compatible with the Hodge decomposition. We have ranks

($V_- = \overline{V_+}$; Q pairs V_+ & V_-)

	V_+	V_-
$(1,0)$	2	1
$(0,1)$	1	2

and a $\mathbb{Q}(\zeta_3)$-Hermitian form $h(\cdot,\cdot) := \sqrt{-3}\, Q(\cdot, \bar{\cdot})$ on V_+ with signature $(2,1)$. The resulting Hodge structures are parametrized by a 2-ball $\mathbb{B}_2 \subset \mathfrak{h}_3$, with generic MT group M_φ having $M_\varphi(\mathbb{R}) \doteq U(2,1)$. The family of Jacobians of these curves constitute an example of the families arising in Shimura's work.

The general definition of MTG (for HS of any weight) came a few years after Mumford's, in a paper of Deligne on the Weil conjectures for K3 surfaces.