3. The theorem of Hazama & Murty

So what about algebraic cycles? The point is that MT groups pick out the classes that should come from them, according to the Hodge Conjecture (HC).

In this section we look at how MT groups lead to a proof of the HC for a class of abelian varieties, basically by showing there are "no interesting Hodge classes".

Let A = abelian d-fold, $V = H^1(A)$, with φ corresponding to its Hodge decomposition and Q a polarization (i.e. corr. to a Riemann form). Write

$$M(A) = M_\varphi, \quad L(A) = L_\varphi \qquad (\text{in } G = Sp(V, Q))$$

and $\mathcal{E}(A) = E_\varphi \cong End(A) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. We continue writing $Hg^{k,l}V$ (Hodge tensors) to distinguish it from $Hg^k(A) = H^{2k}(A,\mathbb{Q}) \cap H^{k,k}(A)$. Note that $Hg^k(A) \subset \Lambda^{2k}V \ (= H^{2k}(A,\mathbb{Q}))$, and set

$$(1) \quad \begin{cases} Hg(A) := \bigoplus_{k=0}^{d} Hg^k(A) = Hg\,V \cap H^*(A) \\ D(A) := \langle Hg^1(A) \rangle \subseteq Hg(A) \end{cases}.$$

By Lefschetz (1,1), $D(A)$ = the classes generated by (intersection products of) divisors, and so "trivially"

(2) $\quad$ HC holds for A if $D(A) = Hg(A)$.

If $D(A) \subsetneq Hg(A)$, the classes in the complement are exceptional Hodge classes, and we want to look for situations where there aren't any.

Now $Hg(A) = H^*(A)^{M(A)} = (\Lambda^* V)^{M(A)}$, while $D(A) \subseteq H^*(A)^{L(A)}$ since $L(A)$ fixes $Hg^{2,0}V$. If A is nondegenerate ($\underset{\text{def.}}{\Leftrightarrow}$ (V, φ) is), then $L(A) = M(A)$ and so the main issue is:

in $H^*(A)$, does $L(A)$ fix only $D(A)$?

This is a question in invariant theory: Sometimes, if you ask to fix one set of tensors, the group this cuts out fixes many more; to prove HC we need to show this doesn't happen! To have a chance, we need to identify the simple factors of $L(A)$.

So assume first A simple, so that $\mathcal{E}(A) =: \mathcal{D}$ is a division algebra. Let $E \subset \mathcal{D}$ be a maximal subfield; by the Albert classification we have either:

(I) $E = \mathcal{D} =$ totally real field F

(II/III) $\mathcal{D}$ quaternion algebra / totally real field F $\Rightarrow [E:F] = 2$, E CM

(IV) $\mathcal{D}$ division algebra / CM field F, $\begin{cases} [E:F] = q \\ [\mathcal{D}:F] = q^2 \end{cases}$, E CM

[Note: $\mathcal{D}$ commutes with F but not E.]

Let $\begin{cases} F^+ \\ E^+ \end{cases}$ be a maximal totally real subfield of $\begin{cases} F \\ E \end{cases}$.

[Remark: (I) $F^+ = F = E^+ = E$; (II/III) $F^+ = F = E^+$ index 2 in E;
(IV) $[E:E^+] = 2 = [F:F^+]$.]

Lemma 1 (Shimura*): $\exists\,!$ F^+-bilinear form $\mathcal{Q} : V \times V \to D$ s.t.

(i) $Q(v', v) = \mathrm{tr}_{D/\mathbb{Q}}\, \mathcal{Q}(v', v)$

(ii) $\mathcal{Q}(\delta' v', \delta v) = \delta' \mathcal{Q}(v', v)\, \delta^\dagger$

(iii) $\mathcal{Q}(v', v) = -\mathcal{Q}(v, v')^\dagger$

(where $D \xrightarrow{\dagger} D^{op}$ is the Rosati involution of §I.B.2 p.11).

I'll give the flavor of proof of this later in a special case.

Now writing $V_{\mathbb{R}} = \bigoplus_{\rho \in \mathrm{Hom}(F^+, \mathbb{R})} V_\rho$ for the eigendecomposition under F^+ ($\dim_{\mathbb{R}} V_\rho = 2d_\rho$, $\sum d_\rho = d$),

$F^+ \subset \mathcal{E}(A) \implies$ { Hodge decomposition, action of $M(A)_{\mathbb{R}}$, action of $L(A)_{\mathbb{R}}$ } are compatible with $\bigoplus_\rho$.

Moreover

- $M(A)_{\mathbb{R}} = \prod_\rho M_\rho \underset{M \text{ respects } Q}{\implies} Q_{\mathbb{R}} = \boxtimes\, Q_\rho$;
- $L(A)_{\mathbb{R}} = \prod_\rho L_\rho \leq \prod_\rho \underbrace{Sp(V_\rho, Q_\rho)}_{G_\rho}$; and

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* Shimura, "On the field of definition of a field of automorphic functions".

• $$D_{\mathbb{R}} = \prod D_\rho = \begin{cases} \mathbb{R} & (\mathrm{I}) \\ M_2(\mathbb{R}) & (\mathrm{II}) \\ \mathbb{H} & (\mathrm{III}) \\ M_q(\mathbb{C}) & (\mathrm{IV}) \end{cases}$$

$$\Longrightarrow L_\rho = \begin{cases} (\mathrm{I}) & G_\rho = Sp_{2d_\rho}(\mathbb{R}), \quad V_\rho = st & (q=1) \\ (\mathrm{II}) & G_\rho^{M_2(\mathbb{R})} \cong Sp_{d_\rho}(\mathbb{R}), \\ (\mathrm{III}) & G_\rho^{\mathbb{H}} \cong SO^*(d_\rho), \end{cases} \Big\} \; V_\rho = st^{\oplus 2} \quad (q=2)$$

$$(\mathrm{IV}) \quad \underbrace{G_\rho^{M_q(\mathbb{C})} \cong U\left(p_\rho, \tfrac{d_\rho}{q} - p_\rho\right), \quad V_\rho = Res_{\mathbb{C}/\mathbb{R}}(st)^{\oplus q}}_{\otimes \mathbb{C}:\ GL(d_\rho/q),\ st^{\oplus q} \oplus st^{*\oplus q}} \qquad q \geq 1$$

In the non-simple case, applying Poincaré complete reducibility (or semisimplicity of PHS) to A gives

$$A \underset{isog.}{\cong} B_1^{\times n_1} \times \cdots \times B_r^{\times n_r} \qquad (B_i = \text{simple, non-isogenous})$$

$$\Longrightarrow \begin{cases} V = \bigoplus_i V_i^{n_i}, \quad V_i = H^1(B_i) \\ \mathcal{E}(A) = \prod_i \mathcal{E}(B_i^{n_i}) = \prod M_{n_i}(\mathcal{E}(B_i)) \end{cases}$$

(any auto. of $V_i^{n_i}$ acts the same on all copies of V_i)

$$\Longrightarrow L(A) = \prod_i L(B_i^{n_i}) = \prod_i L(B_i)$$

We now have enough info to prove

__Theorem 1 (Hazama, Murty^{1984}):__ Let A be an abelian variety. __If__ A is nondegenerate, and no simple factor of $\mathcal{E}(A)$ is of Albert type (III), then the HC holds for all powers of A.

More precisely,

(3) $\mathrm{Hg}(A^k) = \mathcal{D}(A^k)$ $(\forall k \geq 1)$ $\Longleftarrow$ A has no factors of type (III) and $M(A) = L(A)$ $(\ast)$

Proof:

($\Leftarrow$) Here's the basic idea/calculation:

$$\mathrm{Hg}(A^k) = H^*(A^k, \mathbb{Q})^{M(A)} \overset{\text{nondegeneracy } (M(A)=L(A))}{=} H^*(A^k, \mathbb{Q})^{L(A)}$$

$$= \bigotimes_{i=1}^{r} H^*(B_i^{kn_i}, \mathbb{Q})^{L(B_i)} \quad \left[\text{since } (V \otimes W)^{A \times B} = V^A \otimes W^B \text{ in general}\right]$$

(4) $$\overset{?}{=} \bigotimes_{i=1}^{r} D(B_i^{kn_i})$$

$$= D(A^k) \quad \left[\text{since } D(B_1^k \times B_2^l) = D(B_1^k) \otimes D(B_2^l)\right]$$

The sticky point is (4), which isn't always true. We must show that if B is simple of type I, II, or IV, then

(5) $$H^*(B^n, \mathbb{Q})^{L(B)} = D(B^n). \quad (\forall n)$$

Put $V = H^1(B)$, so that $H^1(B^n) = V^{\oplus n}$ and $H^2(B^n) = \Lambda^2 V^{\oplus n}$ = copies of $\Lambda^2 V$ and $V^{\otimes 2}$. But $L(B)$ fixes $\mathrm{Hg}^{1,1} V \Rightarrow$ fixes $\mathrm{Hg}^{2,0} V \Rightarrow$ fixes $\mathrm{Hg}^1(B^n) = D(B^n)$. So

$$D(B^n) \subseteq H^2(B^n, \mathbb{Q})^{L(B)} \overset{(M \leq L)}{\subseteq} H^2(B^n, \mathbb{Q})^{M(B)} = \mathrm{Hg}^1(B^n) = D^1(B^n),$$

and (5) holds "for $\ast = 2$".

Now we have $D(B^n) \subseteq H^*(B^n)^{L(B)}$; we need the reverse. But

$$H^*(B^n, \mathbb{R})^{L(B)_{\mathbb{R}}} \cong \bigotimes_{\rho} \left(\Lambda^* V_\rho^{\oplus n}\right)^{L_\rho}$$

where V_ρ is as above. A result of Weyl (in invariant theory over $\mathbb{C}$) [this is what the unitary groups become] then shows that invariants of GL and Sp (but not SO) are generated in degree $* = 2$. (See [FH] appendix F, Prop. F.13 and Exercise F.20.) Since we already proved (5) for $* = 2$, we're done.

($\Longrightarrow$) This is now easy: again by Weyl, the invariant tensors of $SO(2m)$ are generated by the (degree 2) orthogonal form and the determinant Δ, which has degree $2m \geq 4$, and cannot be written as a polynomial in the form. So if any B_i has type III, you get a determinant Hodge cycle which is exceptional.

If A is degenerate (which can happen even if the B_i aren't), then because $L(A) \supsetneq M(A)$ are characterized by their fixed tensors, and $\bigoplus_k H^*(A^k)$ contains all tensor spaces, we cannot have $D(A^k) = Hg(A^k)$ for all k (though it can happen for $k=1$).

□

The next result gives some concrete examples where the condition (*) in the theorem holds. [Note: Let K be an imaginary quadratic field, A abelian d-fold w/ K-multiplication ($K \hookrightarrow E(A)$). If the eigenspace dims. for the action of K on $H^{1,0}(A)$ are (a,b), we say the K-mult. has type (a,b).]

Theorem 2 (Ribet / Tankeev): Let A be an abelian d-fold. Then A is nondegenerate (and the HC holds for all A^k) if

(i) $\mathcal{E}(A) = F$ totally real s.t. $d/[F:\mathbb{Q}]$ is odd

(ii) $\mathcal{E}(A) = K$ imaginary quadratic w/type (a,b), a & b relatively prime

or

(iii) d prime and $\mathcal{E}(A) = F$ CM of degree $2d$.

Corollary: Let A be a simple abelian variety of prime dimension d. Then A is nondegenerate, and the HC holds for all A^k.

Proof (Corollary): A can't be of type II or III, unless $d=2$ and the type is II:* $\dim A = 2[F:\mathbb{Q}]\dim_D V$ shows that d would have to be even. So either $\mathcal{E}(A) = \mathbb{Q}$, F totally real of deg. d (i), or E CM of degree 2 (ii) or $2d$ (iii). Apply Thms. 1 & 2. □

(Note: $[F:\mathbb{Q}]$ must divide $\dim H^{1,0} = d$ (why?))

For the (sketch of) proof of Thm. 2, we need

Lemma 2**: Let V_1, V_2 be representations of $\mathfrak{g}_1, \mathfrak{g}_2$ resp., $\mathfrak{g} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (simple Lie algebras $/\mathbb{C}$). Assume that under both projections, $\mathfrak{g} \twoheadrightarrow \mathfrak{g}_1, \mathfrak{g}_2$. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ OR $\mathfrak{g}$ is the graph of an isomorphism $\mathfrak{g}_1 \cong \mathfrak{g}_2$

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* See the analysis of the $d=2$ case in Moonen's article, or below.

** Proof: Exercise. Ditto for Lemma 3.

<u>Lemma 3</u>: Let $\mathfrak{g} \subset \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ be a subalgebra, $\mathfrak{g}_i$ simple $/\mathbb{C}$.
Assume all projections $\mathfrak{g} \to \mathfrak{g}_i$ and $\mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$ $(i<j)$ are surjective.

Then $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$.

<u>Sketch of pf. of Thm. 2(i)</u>: $\underset{\text{tot. real}}{F} \xrightarrow[\cong]{\alpha} \mathcal{E}(A) \Rightarrow \underset{(\text{Rosati})}{\dagger} = id$

$\Rightarrow Q(\varepsilon v, w) = Q(v, \varepsilon w) \Longrightarrow V_F = \bigoplus_\rho V_\rho$ and $\left(\begin{matrix}\text{if } F = \mathbb{Q}(f),\ \varepsilon = \alpha(f), \\ v_\rho \in V_\rho\end{matrix}\right)$

$$\rho(f) Q(v_\rho, v_{\rho'}) = Q(\varepsilon(f) v_\rho, v_{\rho'}) = Q(v_\rho, \varepsilon(f) v_{\rho'})$$
$$= \rho'(f) Q(v_\rho, v_{\rho'}) \Rightarrow \underline{Q(v_\rho, v_{\rho'}) = 0}$$

i.e. Q_F restricts to $Q_\rho : V_\rho \times V_\rho \to F(V_\rho)$

$\Rightarrow (V, Q) = \mathrm{Res}_{F/\mathbb{Q}}(V_{\rho_0}, Q_{\rho_0})$ (for one fixed ρ_0)*

<u>OR</u> writing $v = \sum v_\rho$ etc. and taking $\mathcal{Q}(v, w) := Q_{\rho_0}(v_{\rho_0}, w_{\rho_0})$
we have $\mathrm{Tr}_{F/\mathbb{Q}} Q(v_{\rho_0}, w_{\rho_0}) = \sum_\rho Q(v_\rho, w_\rho) = Q(v, w)$
hence $\mathcal{Q} : V \times V \to F$ s.t. $\mathrm{Tr}_{F/\mathbb{Q}} \mathcal{Q} = Q$.

If $\underset{\sim\sim}{M = L}$ $\left(= \mathrm{Res}_{F/\mathbb{Q}}(\mathfrak{sp}(V_\rho, Q_\rho))\right)$ then we are done by Theorem 1.

First note that $Z(M) \subset \mathrm{End}_M(V) = F$ and $Z(M) \subset L \underset{(\text{why?})}{\Longrightarrow}$
$Z(M)$ finite $\Rightarrow$ M semisimple. So write $M_\mathbb{C} = \prod_i M_i$, M_i simple $/\mathbb{C}$.

Also, $\left\{\begin{matrix} \bullet V \text{ irrep. of } M \text{ (otherwise we have an extra projection endomorphism and } L \text{ would be smaller)} \\ \bullet \text{each } V_{\rho,\mathbb{C}} \text{ is a representation of } M_\mathbb{C} \text{ (since } M \text{ commutes with } F) \end{matrix}\right.$

$\Rightarrow \underset{\text{is an irrep. of } M_\mathbb{C}}{V_{\rho,\mathbb{C}} = \bigotimes_i V_\rho^i}$, and each V_ρ^i is a (poss. trivial) <u>irrep</u>. of M_i.

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* Often with $\mathrm{Res}_{F/\mathbb{Q}}$ one would think of V as a vect. space over F <u>rather</u> than writing V_{ρ_0}. But $V_F = V \otimes_\mathbb{Q} F = \bigoplus V_\rho$ and as F-vector space, V_ρ is $\cong$ to "V viewed as v.s./F", and for some purposes is less confusing.

By root-weight theory, one deduces (basically by the theory of "symplectic nodes"): that the M_i are symplectic or orthogonal, of type $D_{\ell \geq 4}$ in the latter case; and that the V_ρ^i are either minuscule ($\Rightarrow$ standard) or trivial. But

$$\frac{d}{[F:\mathbb{Q}]} \text{ odd} \Rightarrow \dim_F V_\rho = 2 \cdot \text{odd} \Rightarrow 4 \nmid \dim_{\mathbb{C}} V_\rho^i \Rightarrow \text{no } D_\ell\text{'s}$$

$\Downarrow$

only one V_ρ^i of even dim (for each ρ) $\overset{\text{symp. standard or trivial}}{\Rightarrow}$ only one (symplectic) standard (for each ρ).

$\Rightarrow$ image of $M_{\mathbb{C}}$ in $GL(V_{\rho,\mathbb{C}})$ must be the symplectic group of (V_ρ, some alternating form)

but since it is already contained in $Sp(V_\rho, Q_\rho)$, this *is* the image

$$\Rightarrow \quad M_{\mathbb{C}} \leq L_{\mathbb{C}} = \prod_\rho \overbrace{Sp(V_\rho, Q_\rho)}^{G_\rho}$$

and $M_{\mathbb{C}}$ projects ONTO each factor.

Now since $\mathrm{End}_M V = F$, the V_ρ are pairwise $\not\cong$ as $M_{\mathbb{C}}$-modules. Consider the projection to a product $G_{\rho_i} \times G_{\rho_j}$ — call this M_{ij}. Clearly $M_{ij} \twoheadrightarrow G_{\rho_i}, G_{\rho_j}$. If $G_{\rho_i} \overset{\cong}{\twoheadleftarrow} M_{ij} \overset{\cong}{\twoheadrightarrow} G_{\rho_j}$ then because the G's are full symplectic groups, this implies V_{ρ_i} and V_{ρ_j} are isomorphic representations of M_{ij},* hence of $M_{\mathbb{C}}$, a contradiction! So by Lemma 2, $M_{ij} = G_{\rho_i} \times G_{\rho_j}$. Applying Lemma 3,

$$M_{\mathbb{C}} = \prod_\rho G_\rho = L_{\mathbb{C}} \quad \text{as desired.}$$

□

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* A priori they are only isomorphic *modulo* automorphisms of M_{ij}, but for a symplectic group these are all inner and it doesn't matter.

To conclude this section, we consider the possible Mumford-Tate groups for simple abelian varieties of dimensions 1 thru 4. From Example 2.5 we have:

(d = 1) ⊛

(Albert) type	$\mathcal{E}$	$L_{\mathbb{R}} (= M_{\mathbb{R}})$
I	$\mathbb{Q}$	SL_2
IV	F quadratic	$U(1)$ ("$\cong S^1$")

while from Theorem 2 (+ Corollary) above:

(d = 3)

type	$\mathcal{E}$	$L_{\mathbb{R}} (= M_{\mathbb{R}})$
I	$\mathbb{Q}$	Sp_6
	F cubic	$SL_2^{\times 3}$
IV	F quadratic	$U(2,1)$
	F deg. 6	$U(1)^{\times 3}$

For $d=2$, the following observation is helpful. Since the endomorphisms of V commuting with M belong to $\mathcal{E}^*$ (thought of as an algebraic group$/\mathbb{Q}$), the center $Z(M) \leq \mathcal{E}^*$. Since this commutes with $\mathcal{E}^*$, and the center of $\mathcal{E}^*$ is $\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ (this is F^* thought of as $\mathbb{Q}$-algebraic group), in fact $Z(M) \leq \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$. But M preserves the polarization Q, and the Rosati involution is always complex conjugation on fields, so $Q(v,v') = Q(\varepsilon v, \varepsilon v') = Q(\varepsilon\bar{\varepsilon} v, v')$ $(\forall v, v')$ for

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⊛ Note: In general, we shall denote (as above) the center of $\mathcal{E}$ by F. For types I–III, F is totally real; for IV, it is CM (so e.g. "quadratic/type IV" $\Rightarrow$ imaginary quadratic).

$\varepsilon \in Z(M) \implies \varepsilon\bar{\varepsilon} = id_V$; it follows that $Z(M)$ belongs to the unitary group U_F (norm 1 elements of F^*, viewed as $\mathbb{Q}$-alg gp.). Since this is <u>finite</u> when F is totally real, we have

<u>Lemma 4</u> (Tankeev): If A is simple of type $\neq$ IV (or more generally A has no factor of type IV), then $M(A)$ is semisimple.

Now consider the case $d = 2$:

- if (V, φ) has endomorphisms by a <u>definite</u> quaternion algebra $D/\mathbb{Q}$, then we are in type III and M is semisimple. But also: $M \leq Aut(V,Q)^{D} \cong SU^*(2) =$ TORUS! $\leadsto$ contradiction.
- if (V, φ) has endomorphisms by an imaginary quadratic field F, then $M \leq SU(1,1)$, which is isomorphic to $SL_2(\mathbb{R})$ (by the "Cayley transform"). This basically forces A to break into 2 isomorphic elliptic curves, so in fact E is bigger than F.

So we are left with:

$(d=2)$ type	$\mathcal{E}$	$L_{\mathbb{R}} (= M_{\mathbb{R}})$
I	$\mathbb{Q}$	Sp_4
	F quadratic	$SL_2 \times SL_2$
II	$D/\mathbb{Q}$	SL_2
IV	F quartic	$U(1)^{\times 2}$

For the cases "$\mathcal{E} = F$", $L = M$ is covered by Theorem 2. The other two use Lemma 4: e.g. for $\mathcal{E} = D$, $Aut(V,Q)^{D} \cong SL_2$; $\therefore$

M were any smaller it would be a torus (hence not semisimple).

Finally, for $d=4$ we get our first instance of both degeneracy and type II:

(d = 4)

type	$\mathcal{E}$	$L_{\mathbb{R}}$	$M_{\mathbb{R}}$		exceptional Hodge classes
I	$\mathbb{Q}$	Sp_8	Sp_8		
			$SL_2^{\times 3}$	$\longrightarrow$	on self-product only (Mumford)
	F quadratic	$Sp_4^{\times 2}$	$Sp_4^{\times 2}$		
	F quartic	$SL_2^{\times 4}$	$SL_2^{\times 4}$		
II	$\mathcal{D}/\mathbb{Q}$	Sp_4	Sp_4		
	$\mathcal{D}$/quadratic F	$SL_2^{\times 2}$	$SL_2^{\times 2}$		
III	$\mathcal{D}/\mathbb{Q}$	$SO^*(4)$	$SO^*(4)$	$\longrightarrow$	determinant Hodge class (Murty)
IV	F quadratic	$U(1,3)$	$U(1,3)$		
		$U(2,2)$	$SU(2,2)$	$\longrightarrow$	Weil-Hodge class (Weil)
	F quartic	$SL_2^{\times 2}$	$SL_2^{\times 2}$		
(CM cases)	F deg. 8	$U(1)^{\times 4}$	$U(1)^{\times 4}$		
(CM cases)			$U(1)^{\times 3}$	$\longrightarrow$	first example of exc. Hodge class (Mumford)

In fact, the Hodge classes (of type $(2,2)$ in $H^4(A)$) in the last three cases are all "Weil-Hodge", which will be defined in §I.B.5.

[Notes: I added notes by Wahanah on classical groups to the resources for §I.B.2. He has an explanation of the "quaternionic" groups with some good exercises on pp. 8-12.]