4. Mumford-Tate domains and period mappings

Given $V, Q, n,$ (*) let $\underline{h} := (h^{p,n-p})_{p\in\mathbb{Z}}$ satisfy $h^{p,n-p} = h^{n-p,p}$ and $\sum_p h^{p,n-p} = \dim V$. The period domain

(1) $$D_{\underline{h}} := \left\{ \varphi \;\middle|\; \begin{array}{l}(V,\varphi,Q) \text{ is a PHS of weight } n, \\ \text{with } \dim V_\varphi^{p,n-p} = h^{p,n-p}(V_\varphi)\end{array} \right\}$$

is a real-analytic open subset in its compact dual

(2) $$\check{D}_{\underline{h}} := \left\{ F^\bullet \;\middle|\; \begin{array}{l}F^\bullet \text{ is a flag on } V \text{ satisfying } (HR_1) \\ \text{and } \dim(F^p/F^{p+1}) = h^{p,n-p}\end{array} \right\}$$

[where (HR_1) for flags is: $Q(F^p, F^{p'}) = 0$ for $p+p' > n$], a complex projective variety. Writing $G = \mathrm{Aut}(V,Q)$,

(3) $$g.\varphi := g\varphi g^{-1} \in D_{\underline{h}}$$

defines an action of $G(\mathbb{R})$ on $D_{\underline{h}}$.

Proposition 1: (i) $D_{\underline{h}} \cong G(\mathbb{R}).\varphi \cong G(\mathbb{R})/\mathcal{H}_\varphi$, where the isotropy group $\mathcal{H}_\varphi$ is compact.

(ii) $\check{D}_{\underline{h}} \cong G(\mathbb{C}).F^\bullet_\varphi \cong G(\mathbb{C})/P_{F^\bullet_\varphi}$, where $P_{F^\bullet_\varphi}$ is parabolic.

Proof: Exercise in bilinear forms using the (HR_1) & (HR_2). □

We have the identification $T_\varphi D_{\underline{h}} \cong \underset{\mathrm{Lie}(G)}{\mathfrak{g}}/\underset{\mathrm{Lie}(\mathcal{H}_\varphi)}{\mathfrak{h}_\varphi} \cong \bigoplus_{i<0} \mathfrak{g}^{(+i,-i)}$, where $\mathfrak{g}_{\mathbb{C}} = \bigoplus_i \mathfrak{g}^{(i,-i)}$ under $\mathrm{Ad}\circ\varphi$.

(*) as usual, V is a $\mathbb{Q}$-vector space, $n \in \mathbb{Z}$, $Q: V\times V \to \mathbb{Q}$ nondegenerate bilinear form, "$(-1)^n$-symmetric"

Example 1

- $n = 2m+1$ odd $\rightsquigarrow$ $D_{\underline{h}} = \overbrace{Sp_n(\mathbb{R})}^{G} \Big/ \prod_{p \le m} U(h^{p,n-p})$
- $n = 2m$ even $\rightsquigarrow$ $D_{\underline{h}} = \overbrace{SO(h_{odd}, h_{even})}^{G} \Big/ SO(h^{m,m}) \times \prod_{p<m} U(h^{p,n-p})$
- $D_{\underline{h}}$ Hermitian symmetric $\overset{\text{def.}}{\iff}$ $\varphi(i)$ induces a "symmetry" $s_\varphi : D_{\underline{h}} \to D_{\underline{h}}$ with $s_\varphi^2 = id$, having $\varphi \in D_{\underline{h}}$ as isolated fixed point

(4) $\iff$ the eigenvalues of $Ad \circ \varphi(z)$ on $\mathfrak{g} = Lie(G)$ are of the form $\ldots, z^{-6}, z^{-2}, 1, z^2, z^6, \ldots$ (i.e. $\mathfrak{g}^{(i,-i)} = \{0\}$ for $i \neq 0$ even)

(5) $\iff$ $\mathcal{H}_\varphi$ is a maximal compact subgroup of $G(\mathbb{R})$

This can happen when $G(\mathbb{R}) = SO(2,n)$ or $Sp_n(\mathbb{R})$, and the basic examples are $\underline{h} = (a,a)$ and $(1,b,1)$. (The others have "gaps" in the list of Hodge numbers.) //

The Mumford–Tate domains are the subsets $D \subset D_{\underline{h}}$ obtained by choosing $\varphi \in D_{\underline{h}}$ and taking the orbit

$$D := M_\varphi(\mathbb{R}).\varphi \cong M_\varphi(\mathbb{R})/H_\varphi \qquad (\text{M-T domain})$$

(6) $$\cap$$

$$\check{D} := M_\varphi(\mathbb{C}).F^\bullet_\varphi \cong M_\varphi(\mathbb{C})/P_{F^\bullet_\varphi} . \qquad (\text{compact dual}) = \text{proj. variety}/\bar{\mathbb{Q}}$$

This produces homogeneous spaces of much greater variety, including Hermitian symmetric domains of type A_n, E_6 and E_7 (e.g. the Poincaré 2-ball is type A_2) and ones parametrizing (gap-free) Hodge structures of level/weight > 2. Note that (4) & (5) above remain valid for this more general case (replacing G by M & $\mathfrak{g}$ by $\mathfrak{m}$).

Example 2/ For easy examples in the spirit of the Poincaré 2-ball (Picard curves), one can start with $\varphi \in D_{\underline{h}}$ compatible with a cubic automorphism of V (and $V_F = V_+ \oplus V_-$ as before). When $\underline{h} = (1, 2n, 1)$ [resp. $\underline{h} = (n+1, n+1)$] and $\underline{h}_+ = (0, n, 1)$ [resp. $(n, 1)$] this yields embeddings of the n-ball

$$D \cong \mathbb{B}_n \cong U(1,n) / \{U(n) \times U(1)\}$$

into type IV [resp. III] Hermitian-symmetric $D_{\underline{h}}$. The same story with $\underline{h} = (1, n, n, 1)$ and $\underline{h}_+ = (1, n, 0, 0)$ embeds $\mathbb{B}_n \hookrightarrow$ non-HSD $D_{\underline{h}}$. //

Let $\mathcal{V} = (\mathbb{V}, \mathcal{V}, \mathcal{F}^\bullet)$ be a holomorphic family of pure HS over a complex manifold $\mathcal{S}$ ($\mathbb{V} =$ $\mathbb{Q}$-local system; $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_{\mathcal{S}}$ = [sheaf of sections of] holo. vector bundle; $\mathcal{F}^\bullet$ = filtration by [sheaves of sections of] holomorphic subbundles). Take

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{\mathcal{S}}$$

to be the flat connection with $\nabla(\mathbb{V}) = 0$, $\tilde{\mathcal{S}} \xrightarrow{\pi} \mathcal{S}$ the universal cover, and $\tilde{\mathcal{V}} := \pi^* \mathcal{V}$. Fix $s_0 = \pi(\tilde{s}_0) \in \mathcal{S}$, set $V := \mathbb{V}|_{s_0}$, so that $\tilde{\mathbb{V}} = V \otimes \mathbb{Z}_{\tilde{\mathcal{S}}}$. Given $t \in T^{m,n} V$ a Hodge (p,p)-tensor at $\tilde{s}_0$,

$$\tilde{\mathcal{S}}(t) := \{ \tilde{s} \in \tilde{\mathcal{S}} \mid t \in F^p_{\tilde{s}} \} \subset \tilde{\mathcal{S}}$$

is an analytic subvariety, and so also is $\mathcal{S}(t) := \pi(\tilde{\mathcal{S}}(t)) \subset \mathcal{S}$, the Hodge locus of t. Assume s_0 avoids $\mathcal{S}_{\mathrm{Hdg}} := \bigcup_{t : \mathcal{S}(t) \neq \mathcal{S}} \mathcal{S}(t)$, so that any t' Hodge at s_0 is Hodge everywhere. So we have

Proposition 2 (Deligne): Let M_s denote the M-T group of V_s. Then M_s is locally constant $(=: M)$ off a countable union of proper analytic subvarieties, and $\leq M$ everywhere.

Now assume $\mathcal{V}$ is a polarized VHS (variation of HS), i.e. that

$$\nabla(\mathcal{F}^\bullet) \subset \mathcal{F}^{\bullet-1} \otimes \Omega^1_{\mathcal{S}}. \quad \text{(Griffiths transversality)}$$

Let

$$(7) \qquad \Phi_{\underline{h}} : \mathcal{S} \longrightarrow \Gamma \backslash D_{\underline{h}}$$

be the associated period map. By Prop. 2, the $Hg^{m,n}(V_s)$ are invariant under ∇-flat continuation over $\mathcal{S} \backslash \mathcal{S}_{Hg}$. Since $Q>0$ on Hodge tensors, monodromy

$$(8) \qquad \rho : \pi_1(\mathcal{S}, s_0) \to Aut(V, Q)$$

acts through an $SO_N(\mathbb{Z})$ on each $Hg^{m,n}(V_{s_0})$, which is to say by a finite group. This proves the first part of

Proposition 3 (Deligne, André): (a) The geometric monodromy group

$$(9) \qquad \Pi := \Big(\overline{\rho(\pi_1(\mathcal{S}))}^{Zar}\Big)^{\circ}$$

← identity connected component; Zar ← $\mathbb{Q}$-Zariski closure

is a subgroup of M. (b) $\Pi \trianglelefteq M^{der} := [M, M]$, with equality iff $\mathcal{V}$ has a CM point (i.e., some V_s has abelian MTG).

Proof (of $\Pi \trianglelefteq M$): By Schmid's Theorem of the Fixed Part, $(T^{m,n}\mathbb{V})^{\Pi}$ underlies a sub-HS of $T^{m,n}\mathbb{V}$. Sub-HS are stabilized by M, hence every $g\Pi g^{-1}$ $(g \in M(\mathbb{Q}))$ fixes $(T^{m,n}\mathbb{V})^{\Pi}$. But a subgroup of $GL(V)$ is determined by its fixed tensors (Chevalley), and so every conjugate $g\Pi g^{-1} \leq \Pi$. □

Remark 1: Since M^{der} is semisimple, Prop. 3 $\Rightarrow$ Π is semisimple. //

Remark 2: $\Pi^\circ \leq M \Rightarrow \Phi_{\underline{h}}$ factors thru

$$(10) \qquad \Phi : \mathcal{S} \to \Gamma\backslash D = \Gamma\backslash G(\mathbb{R})/H,$$

after possibly pulling back to a finite cover of $\mathcal{S}$ (to account for the group Π/Π°). By Griffiths-Robles-Toledo, $\Gamma\backslash D$ is ~~al~~ algebraic iff D fibres holomorphically (or antiholomorphically) over a HSD.(*) In higher weight/level, D could be a HSD while $D_{\underline{h}}$ isn't, so (10) could be algebraic while (7) isn't. Hence the value of Proposition 3! //

Now suppose given a PHS (V, φ, Q) with MTG $M \leq G = Aut(V,Q)$; write $D = M(\mathbb{R}).\varphi = M(\mathbb{R})/H$ for the domain, and $Ad : M \twoheadrightarrow M^{ad} \leq Aut(\mathfrak{m}, B)$ (B = Killing form $B(X,Y) = Tr(\mathrm{ad}X \circ \mathrm{ad}Y)$) for the adjoint homomorphism. Then φ induces a HS of weight 0 on the $\mathbb{Q}$-vector spaces $\mathfrak{m} = T_e M$ and $\mathfrak{m}^{ad} = T_e M^{ad}$. Replacing M, V, φ, Q by $M^{ad}, \mathfrak{m}^{ad}, Ad \circ \varphi, -B$ leaves the connected M-T domain $D^\circ = M(\mathbb{R})^\circ/H$ unchanged. This motivates the slightly cheaper question

(11) Which $\mathbb{Q}$-simple adjoint groups are M-T?

Proposition 5 [GGK]: Assume M is $\mathbb{Q}$-simple & adjoint. Then

M is a M-T group $\Longleftrightarrow$ $M(\mathbb{R})$ has a compact maximal torus.

(*) If D is a Hermitian symmetric domain, then $\Gamma\backslash D$ is a connected Shimura variety or locally symmetric variety, but see below.

Proof of ($\Leftarrow$): ($\Rightarrow$ is easy) Exercise! Let $m_{\mathbb{R}} = k \oplus p$ be a Cartan decomp. with $k \supset t_{\mathbb{R}}$, where t = Lie algebra of max. torus.

Let $\Delta = \Delta_c \cup \Delta_n$ = roots of $(m_{\mathbb{C}}, t_{\mathbb{C}})$, R = lattice they generate in $t^*_{\mathbb{C}}$.
(Δ_c compact, Δ_n noncompact)

We have

$$(12) \qquad m_{\mathbb{C}} = t \oplus \Big(\bigoplus_{\alpha\in\Delta_c} m_\alpha\Big) \oplus \Big(\bigoplus_{\beta\in\Delta_n} m_\beta\Big) ;$$

note that $-B(X_\alpha, \overline{X_\alpha}) > 0$ and $-B(X_\beta, \overline{X_\beta}) < 0$.

The (Cartan) involution def'd by $\theta|_k = id_k$, $\theta|_p = -id_p$ is a Lie-algebra homomorphism; so there exists $\Psi : R \to 2\mathbb{Z}/4\mathbb{Z}$ with $\Psi(\alpha) \underset{(4)}{\equiv} 0$ and $\Psi(\beta) \underset{(4)}{\equiv} 2$. Since M is adjoint, $R = \Lambda$ (= weight lattice), which is free (like any lattice). So $\exists$ lift $\tilde{\Psi} : \Lambda \to 2\mathbb{Z}$ of Ψ.

Moreover, the co-character group $X_*(T(\mathbb{C}))$ maps isomorphically to $\mathrm{Hom}(\Lambda, 2\mathbb{Z})$ by $\varphi \mapsto l_\varphi := \frac{d\varphi}{dz}(1)$. So then $\exists$ co-character $\varphi : S^1 \to T(\mathbb{R})$ with $l_\varphi \underset{(4)}{\equiv} \tilde{\Psi}$!

From $Ad(\varphi(z)) X_\alpha = z^{\langle l_\varphi, \alpha\rangle} X_\alpha$, we have:

- $(Ad\circ\varphi)(i) = \theta \Rightarrow -B(\cdot, (Ad\circ\varphi)(i)\overline{(\cdot)}) > 0$ on $m_{\mathbb{C}}$
  $\Rightarrow (m, -B, Ad\circ\varphi)$ is a PHS of weight 0
- $m_{\mathbb{C}} = \bigoplus_j m^{j,-j}$ with

$$(13) \qquad m^{j,-j} = \begin{cases} \bigoplus_{\delta\in\Delta : \langle l_\varphi, \delta\rangle = 2j} m_\delta , & j \neq 0 \\ t \oplus \bigoplus_{\delta\in\Delta : \langle l_\varphi,\delta\rangle = 0} m_\delta , & j = 0 \end{cases}$$

Let $\mathcal{M} \leq M$ be (a) the smallest $\mathbb{Q}$-algebraic subgroup s.t. $Ad\circ m\varphi m^{-1}$ factors thru $\mathcal{M}(\mathbb{R})$ $\forall m \in M(\mathbb{R})$. Equivalently, $\mathcal{M}$ is (b) the MTG of the family $Ad\circ m\varphi m^{-1}$ of HS. By Proposition 2, (b) $\Rightarrow$

$\mathcal{M}$ is the MTG of $\mathrm{Ad} \circ m_0 \varphi m_0^{-1}$ for every m_0 in the complement of a countable union of proper analytic subvarieties (of M). On the other hand, (a) $\Rightarrow \mathcal{M} \trianglelefteq M \underset{M\ \mathbb{Q}\text{-simple}}{\Longrightarrow} \mathcal{M} = M$. Conclude that $M = M_{\mathrm{Ad} \circ m_0 \varphi m_0^{-1}}$ for m_0 as above. $\square$

<u>Remark 3</u>: Prop. 5 "$\Rightarrow$" means that e.g. $SL_{\geq 3}(\mathbb{R})$ can't be M-T.

The proof of "$\Leftarrow$" above leads to a <u>construction</u> of HS with (as MT group) a given $\mathbb{Q}$-simple adjoint group w/ maximal compact torus (including all the split forms of the exceptional groups!). You start with any $\pi : R \to \mathbb{Z}$ with $\pi(\Delta_c) \in 2\mathbb{Z}$, $\pi(\Delta_n) \in 2\mathbb{Z}+1$, then set

$$(14) \qquad \mathfrak{m}^{j,-j} := \begin{cases} \bigoplus_{\delta \in \Delta : \pi(\delta) = j} \mathfrak{m}_\delta, & j \neq 0 \\ \mathfrak{t} \oplus \bigoplus_{\delta \in \Delta : \pi(\delta) = 0} \mathfrak{m}_\delta, & j = 0, \end{cases}$$

and act on this HS by a sufficiently general $\mu \in M(\mathbb{R})$.

To get PHS on other representations than $\mathfrak{m}$, or (say) of odd weight, consider a representation V^λ (of $\mathfrak{m}$, of highest weight λ) defined over $\mathbb{Q}$, and suppose π as above lifts to $\tilde{\pi} : \mathbb{Z}\lambda + R \to \mathbb{Z}$ resp. $\mathbb{Z} + \frac{1}{2}$. Instead of $M = M^{ad}$, let M be the group between M^{sc} & M^{ad} corresponding to $\mathbb{Z}\lambda + R$:

$$\begin{array}{ccccc} \underset{\text{weight lattice}}{\Lambda} & \supseteq & \mathbb{Z}\lambda + R & \supseteq & \underset{\text{root lattice}}{R} \\ \updownarrow & & \updownarrow & & \updownarrow \\ M^{sc} & \twoheadrightarrow & M & \twoheadrightarrow & M^{ad} \end{array}$$

Thus there exists $\varphi: \mathbb{U} \to M$ s.t. $\frac{1}{2} l_\varphi = \tilde{\pi}$, & $\rho: M \hookrightarrow \mathrm{Aut}(V^\lambda, Q)$ (for some symmetric resp. alternating form Q) s.t. $(V^\lambda, \rho\circ\varphi, Q)$ is a PHS of even resp. odd weight. Reasoning as in the above proof, a general conjugate has MTG M.

This is an example of

<u>Definition 1</u>: Let $\mathcal{G}$ be a $\mathbb{Q}$-algebraic group. A <u>Hodge representation</u> of $\mathcal{G}$ is:

- a $\mathbb{Q}$-vector space V,
- a nondegenerate symmetric or alternating ~~bilinear~~ form Q on V,
- a faithful representation $\rho: \mathcal{G} \hookrightarrow \mathrm{Aut}(V, Q)$ (defined $/\mathbb{Q}$), and
- a homomorphism $\varphi: \mathbb{U} \to \mathcal{G}$ (defined $/\mathbb{R}$),

such that $(V, \rho\circ\varphi, Q)$ is a PHS. $\square$

<u>Example 3</u>/ Let A be an (d-dim'l) abelian variety, with PHS $(\underset{V}{\underset{\shortparallel}{H^1(A)}}, \varphi, Q)$, and endomorphisms $\mathcal{E}(A) = \mathrm{End}(A) \otimes \mathbb{Q} = \mathrm{End}_\varphi(V)$. Set

$L := \mathrm{Aut}(V, Q)^{\mathcal{E}(A)} \underset{\rho}{\hookrightarrow} \mathrm{Aut}(V, Q)$, and note that φ factors thru L.

Then $(H^1(A), \varphi, Q, \rho)$ is a Hodge representation of L. //

Given a Hodge rep. of $\mathcal{G}$, of course $\mathcal{G}$ need not be the MT group of the associated PHS. But we can ask whether it is for a generic conjugate. In particular, write M for the M-T group of a very general HS in $\mathcal{G}(\mathbb{R}).\rho\circ\varphi$ ($\equiv \mathcal{G}(\mathbb{R}).\varphi$).

Example 3 (cont'd). In this case, the orbit $L(\mathbb{R}).\varphi$ is a connected component of the locus in $\mathfrak{h}_d$ with endomorphisms by $\mathcal{E}(A)$, where we think of $\mathfrak{h}_d$ as parametrizing abelian varieties of dimension d. Its quotient by an arithmetic $\Gamma \leq L(\mathbb{Q})$ is (essentially) a (connected) PEL Shimura variety. (P = polarization, E = endomorphisms, L = level)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Omega \qquad\qquad\qquad \mathcal{E} \qquad\qquad\qquad \Gamma$ //

Proposition 6: If $\mathcal{G}^{der}$ is simple, then

$$\mathcal{G}^{der} \leq M \leq \mathcal{G}.$$

(In the last example, this allows for $M \subsetneq L$ and for $L(\mathbb{R}).\varphi$ to parametrize degenerate abelian varieties.)

Remark 4: The most general type of (connected) Shimura variety parametrizing abelian varieties is the SV of Hodge type. These all arise by taking an abelian variety A with MT group M and HS φ, and taking the quotient of $\underline{\underline{M(\mathbb{R}).\varphi}}$ by $\Gamma \leq M(\mathbb{Q})$. //

Finally, recall the identification of the tangent space

$$T_\varphi D = \bigoplus_{j<0} \mathfrak{m}_\varphi^{j,-j};$$

in the construction (13) or (14) we easily read off $\dim_{\mathbb{C}} D$ by counting $\sum_{j<0} |\pi^{-1}(j) \cap \Delta|$. For a period mapping (10) associated to a VHS, Griffiths transversality forces $d\Phi$ to lie in the horizontal distribution

$$\mathcal{W}_\varphi = \mathfrak{m}_\varphi^{-1,1} \subset T_\varphi D.$$

($\mathcal{W} \subset TD$ is a subbundle, with fiber $\mathcal{W}_\varphi$ at $\varphi \in D$.)

While D is Hermitian symmetric iff $T_\varphi D = \bigoplus_{\substack{j<0 \\ \text{ODD}}} \mathfrak{m}_\varphi^{j,-j}$,

for defining (connected) <u>Shimura varieties</u> as $\Gamma \backslash D$ one should really insist that $T_\varphi D = \mathfrak{m}_\varphi^{-1,1}$, i.e. that $\underline{TD = \mathcal{W}}$.

This is the case for all the abelian-variety-related examples above.