§5. Weil-Hodge classes and Schoen's construction

We begin with the basic

Example 1. A = abelian 4-fold, $(V = H^1(A), \varphi, Q)$,

$\mathcal{E}(A) \underset{\mu}{\xleftarrow{\cong}} F$ imaginary quadratic (so, type IV)

$\Rightarrow$ get $\underset{\mu(f)}{\overset{\circlearrowright}{V_F}} = \underset{f\cdot \mathrm{Id}}{\overset{\circlearrowright}{V_+}} \oplus \underset{\bar{f}\cdot \mathrm{Id}}{\overset{\circlearrowright}{V_-}}$ compatibly with the Hodge decomposition.

Assume $\dim V_+^{1,0} = \dim V_+^{0,1} = \cdots = 2$, so that the resulting Hermitian form h on V_+ has signature $(2,2)$.

Then φ acts through $L(A) = Sp(V,Q) \cap \mathrm{Res}_{F/\mathbb{Q}}(GL(V_+))$ $\cong U(V_+, h)$. Considering a general HS in the 4-dim'l* Hermitian symmetric domain $D_{2,2} = L(A)(\mathbb{R}).\varphi \cong U(2,2)/U(2)\times U(2)$, we are in the situation of Example 4.3. Applying Proposition 4.6, we have

$$SU(2,2) \leq M(A)(\mathbb{R}) \leq U(2,2). \tag{1}$$

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* In general, since $\dim_{\mathbb{R}} U(p,q) = \dim_{\mathbb{C}} GL(p+q) = (p+q)^2$,

$$\dim_{\mathbb{C}} \underbrace{\left(U(p,q)/U(p)\times U(q)\right)}_{D_{p,q}} = \tfrac{1}{2}\dim_{\mathbb{R}}(D_{p,q}) = \tfrac{1}{2}\{(p+q)^2 - p^2 - q^2\} = pq.$$

[in (1), which is $M(A)$ — SU or U ?]

So which is it? If we let $\{\omega_i\}_{i=1}^4 \subset \Omega^1(A)$ be a basis such that $\omega_1, \omega_2 \in V_+^{1,0}$, then $\mathcal{B}_+ = \{\omega_1, \omega_2, \overline{\omega_3}, \overline{\omega_4}\} \subset V_+$ is a basis. Since

$$[\varphi(z)]_{\mathcal{B}_+} = \mathrm{diag}\{z, z, z^{-1}, z^{-1}\},$$

φ actually factors through $\underline{\underline{S}}U(V_+, h) \cong SU(2,2)$. (Notice that for signature $(1,3)$ this would NOT happen.) So a general A "in" $D_{2,2}$ is degenerate, and has an exceptional Hodge class.

To "see" the Hodge class, notice that $\mu(f)$ acts trivially on

(2) $$\omega_1 \wedge \omega_2 \wedge \overline{\omega_3} \wedge \overline{\omega_4}\ ,\ \omega_3 \wedge \omega_4 \wedge \overline{\omega_1} \wedge \overline{\omega_2} \in H^{2,2}(A).$$

Moreover, these 2 classes span the space $\mathrm{Res}_{F/\mathbb{Q}} \underbrace{\Lambda^4 V_+}_{\text{def'd}/F}$, which is defined $/\mathbb{Q}$. Appropriate linear combinations of (2) therefore give classes in $Hg^2(A)$; as they are not fixed by $L(A) = U(2,2)$, these are the desired exceptional classes. //

More generally, let (V, φ, Q) be a PHS of any weight, with F-multiplication ($F \overset{\mu}{\hookrightarrow} \mathcal{E}_\varphi$, not necessarily imaginary quadratic). Set $h := \dim_F V$, and consider the 1-dimensional F-vector-space $\Lambda_F^h V$, which is in fact a $\mathbb{Q}$-HS of rank $[F:\mathbb{Q}]$. (This is by the compatibility of μ with φ, and because we really mean $\mathrm{Res}_{F/\mathbb{Q}} \Lambda_F^h V$.) The canonical projection

(3) $$p : \Lambda_{(\mathbb{Q})}^h V \twoheadrightarrow \Lambda_F^h V$$

is a morphism of $\mathbb{Q}$-Hodge structures, and so $\ker(p) \subset \Lambda^h V$ is a sub-HS.

<u>Definition 1</u>: The <u>Weil classes</u> associated to (V, φ, Q, μ) are the nonzero elements of the Q-$\perp$-complement W_F to $\ker(p)$ inside $\Lambda^h V$. We have $p|_{W_F} : W_F \xrightarrow{\cong} \Lambda_F^h V$.

<u>WARNING</u>: These are not yet Hodge classes!!

<u>Example 1 (cont'd)</u>/ Let $\{\alpha_i\}_{i=1}^4 \subset V$ be an F-basis. Writing $F = \mathbb{Q}(f)$, we may extend this to a $\mathbb{Q}$-basis by taking $\alpha_{i+4} := \mu(f)\alpha_i$. Λ_F means that we may move $\mu(f)$'s (and $\mu(f)^{-1}$'s, etc.) across the $\wedge$. There are $16 (= 2^4)$ independent classes $\begin{Bmatrix} \alpha_1 \\ \text{or} \\ \alpha_5 \end{Bmatrix} \wedge \begin{Bmatrix} \alpha_2 \\ \text{or} \\ \alpha_6 \end{Bmatrix} \wedge \begin{Bmatrix} \alpha_3 \\ \text{or} \\ \alpha_7 \end{Bmatrix} \wedge \begin{Bmatrix} \alpha_4 \\ \text{or} \\ \alpha_8 \end{Bmatrix} \in \Lambda^4 V$ mapping

to some combination of $\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4$ and $\rho(f)\, \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4$. in $\Lambda^4_F V$. The linear combinations of these 16 will generate $W_F \subset \Lambda^4 V$. //

← (As you can see, the previous approach to this example was a bit simpler. Indeed, it's really only necessary that W_F map $\cong$ to $\Lambda^4_F V$, not that it be $\perp$ to $\ker(\rho)$.)

If $V = H^1(A)$ has F-multiplication for an abelian d-fold A, then $h = 2d/[F:\mathbb{Q}]$, and $W_F \subset \Lambda^h V = H^h(A)$. Also write

$$V_F = \bigoplus_{\rho \in \mathrm{Hom}(F,\mathbb{C})} V_\rho$$

for the decomposition.

Proposition 1 (Moonen–Zarhin): TFAE:

(i) $W_F \cap Hg^{h/2}(A) \neq \{0\}$

(ii) $W_F \subset Hg^{h/2}(A)$

(iii) $\dim V_\rho^{1,0} = \dim V_\rho^{0,1} \quad (\forall \rho)$

(iv) $M(A) \subset \mathrm{Res}_{F/\mathbb{Q}}(SL(V_\rho))$.

(We won't prove this but it should seem plausible from the above examples.)

Definition 2: In this case, W_F is called a space of Weil–Hodge classes for A.

Note that Weil-Hodge classes need not be exceptional: If A is simple of type I and $\mathcal{E}(A) = F$ is real quadratic, then $\underline{\underline{L(A)}}$ satisfies (iv).

The proposition is most effective for producing exceptional Hodge classes on abelian varieties of type $\overline{\text{III}}$ and $\overline{\text{IV}}$.

Unfortunately we really have no idea how to approach the resulting cases of the Hodge conjecture, even in Example 1. There just isn't enough "accessible geometry" in such abelian 4-folds. The only exception to this statement (so far) is for the few families of type (2,2) 4-folds that come from a generalized Prym construction, which must necessarily* have $F = \mathbb{Q}(\zeta_3)$ or $\mathbb{Q}(i)$. We now describe Schoen's proof of the Hodge Conjecture in this case.

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* The point is that because the Prym construction involves a $\mathbb{Z}/m\mathbb{Z}$ étale curve cover, you have to have "multiplication by" $\mathbb{Q}(\zeta_m)$ ($\zeta_m = e^{2\pi i/m}$) in whatever results. But for (2,2) Weil-Hodge classes on an abelian 4-fold, $d = 4 = h \Rightarrow [F:\mathbb{Q}] = 2$. The only quadratic cyclotomic fields are $\mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(i)$, though we naturally wonder if anything can be made of the fact that every $\mathbb{Q}(\sqrt{-D}) \subset \underline{\underline{\text{some}}}\ \mathbb{Q}(\zeta_m)$.

Let

$$C \xrightarrow[m:1]{\pi} X$$

be a cyclic étale cover, with $\mathrm{Aut}(C/X) = \langle \alpha \rangle \ (\cong \mathbb{Z}/m)$;

and

$$\sigma : \mathbb{Z}/m \to \mathrm{Aut}(H^1(C))$$
$$a \longmapsto \sigma(a) := (\alpha^a)^*$$

the induced action on cohomology. We have the characters

$$\chi_r : \mathbb{Z}/m \longrightarrow \mathbb{Q}(\zeta_m)^* \qquad [\zeta_m = e^{2\pi i/m}]$$
$$a \longmapsto \zeta_m^{ar}$$

and the subspaces $H^1(C)^\chi \subset H^1(C, \mathbb{Q}(\zeta_m))$ on which σ acts through χ.

Let $V \subset H^1(C)$ be the $\mathbb{Q}$-subspace with

(4) $$V_{\mathbb{Q}(\zeta_m)} = \sum_{\substack{\chi \\ \text{primitive}}} H^1(C)^\chi \subset H^1(C)_{\mathbb{Q}(\zeta_m)} ;$$

more formally, we could write $V = \mathrm{Res}_{\mathbb{Q}(\zeta_m)/\mathbb{Q}} H^1(C)^\chi$ (for one primitive χ).

Compute

$$h := \text{"} \dim_{\mathbb{Q}(\zeta_m)} V \text{"} = \dim H^1(C)^\chi$$

(Character thry.) $$= \frac{1}{m} \sum_{a \in \mathbb{Z}/m} \chi(-a)\, \mathrm{tr}(\sigma(a))$$

$$= -\sum_{i=0}^{2} (-1)^i \frac{1}{m} \sum_{a \in \mathbb{Z}/m} \chi(-a)\, \mathrm{tr}(\sigma_i(a))$$

$\left\{ \begin{array}{l} \sigma_i : \mathbb{Z}/m \to \mathrm{Aut}(H^i(C)) \\ (\sigma_1 = \sigma) \\ H^0(C)^\chi = \{0\} = H^2(C)^\chi \end{array} \right.$

$$= -\frac{1}{m} \sum_{a \in \mathbb{Z}/m} \chi(-a) \underbrace{\sum_i (-1)^i \mathrm{tr}(\sigma_i(a))}_{=0 \text{ for } a \not\equiv 0}$$

$= 0$ for $a \not\equiv 0$, since α^a acts w/o fixed pts. (apply Lefschetz fixed pt. theorem)

$$= -\frac{1}{m} \cancel{\chi(0)}^{1} \sum_i (-1)^i \underbrace{\mathrm{tr}(\underbrace{\sigma_i(0)}_{id})}_{h^i(C)}$$

(Euler characteristic of C) $$= -\frac{1}{m} e(C)$$

$$= \frac{1}{m}(2g_C - 2)$$

(Riemann-Hurwitz) $\rightarrow$ $= 2g_X - 2$.

Replacing $\{\sigma_i\}_{i=0}^2$ by $\{\tilde{\sigma}_i : \mathbb{Z}/m \to \mathrm{Aut}(H^1(C,\Omega^1_C))\}_{i=0}^1$, essentially the same computation yields $\dim H^{1,0}(C)^\chi = g_X - 1 = \frac{1}{2}\dim H^1(C)^\chi$.

Setting

$$(5) \qquad A := J(V) := \frac{V_{\mathbb{C}}}{F^1V_{\mathbb{C}} + V_{\mathbb{Z}}} \qquad (V_{\mathbb{Z}} = V \cap H^1(C,\mathbb{Z})),$$

$$[\Rightarrow H^1(A) \cong V]$$

we have

$$d := \dim_{\mathbb{C}} A = \frac{1}{2}\dim_{\mathbb{Q}} V = \frac{[\mathbb{Q}(\zeta_m):\mathbb{Q}]\cdot h}{2} = \frac{\phi(m)\,h}{2} = \phi(m)(g_X - 1),$$

where $\phi(m)$ = Euler phi-function. Here are the "lowest-genus" examples of interest, keeping in mind that we want $d \geq 4$ & even:

(6)

m	g_X	g_C	h	d
3	3	7	4	4
4	3	9	4	4
3	4	10	6	6
4	4	13	6	6

(in general, $h \neq d$)

Now consider the action of the symmetric group $\mathfrak{S}_h$ on C^h, by permuting factors. We have

$$H^1(C)^{\otimes h} \subset H^h(C^h)$$

and$^{\circledast}$

$$\wedge^h H^1(C) \cong \left(H^1(C)^{\otimes h}\right)^{\mathfrak{S}_h} \subset H^h(C^h)^{\mathfrak{S}_h}.$$

$\circledast$ This is confusing. Simple example: Consider $dz = \omega$ on $\mathbb{C}$, and the projections $\pi_1, \pi_2 : \mathbb{C}^2_{x,y} \to \mathbb{C}_z$. We have $\pi_1^*\omega \wedge \pi_2^*\omega$ (analogous to form in $H^1(C)\otimes H^1(C) \subset H^2(C^2)$), which is $dx \wedge dy$; this becomes $dy \wedge dx = -dx \wedge dy$ under exchange of factors.

Writing $\varepsilon : (\mathbb{Z}/m)^h \twoheadrightarrow \mathbb{Z}/m$ for the augmentation $(a_1, \ldots, a_h) \mapsto \sum a_i$, let

$$\mathcal{N} := \ker(\varepsilon) \subset (\mathbb{Z}/m)^h =: \tilde{\mathcal{N}}.$$

We have

$$\otimes^h_{\mathbb{Q}(\zeta_m)} V \cong (V^{\otimes h})^{\mathcal{N}} \subset H^h(C^h)^{\mathcal{N}}$$

since acting by (say) $(a, -a, 0, \ldots, 0) \in \mathcal{N}$ sends $v_1 \otimes \sigma(a) v_2 \otimes \cdots$ to $\sigma(a) v_1 \otimes v_2 \otimes \cdots$ (i.e. allows us to move $\mathbb{Q}(\zeta_m)$ across the $\otimes$); hence

(7) $$\Lambda^h_{\mathbb{Q}(\zeta_m)} V \cong (V^{\otimes h})^{\mathcal{N} S_h} =: U \subset H^h(C^h)^{\mathcal{N} S_h}.$$

Since $h = \dim_{\mathbb{Q}(\zeta_m)} V$, U has dimension 1 over $\mathbb{Q}(\zeta_m)$ (dimension $\phi(m)$ over $\mathbb{Q}$); it also clearly identifies with

$$\sum_{\chi} \Lambda^h V^{\chi} = \sum_{\chi \text{ (primitive)}} \Lambda^h H^1(C)^{\chi} \cong \sum_{\chi \text{ (primitive)}} \Lambda^h H^1(A)^{\chi} =: U' \subset H^h(A),$$

a space of Weil classes. That these are Weil-Hodge classes follows from $h^{1,0}(C)^{\chi} = \frac{1}{2} h^1(C)^{\chi}$ $(\Rightarrow \dim(V^{1,0})^{\chi} = \frac{1}{2}\dim(V^{\chi}))$, as observed above, which also implies that $h/2$ is an integer.

Theorem 1 (Schoen): (a) The Hodge Conjecture is true for these Weil-Hodge classes; that is,

(8) $$U' \subset cl\left(CH^{h/2}(A)_{\mathbb{Q}}\right) \subset H_g^{h/2}(A).$$

(b) Ditto for the corresponding classes on C^h:

(9) $$U \subset cl\left(CH^{h/2}(C^h)\right) \subset H_g^{h/2}(C^h).$$

Proof:
We shall first prove (b), then show that (a) follows from it.*

Look at the big diagram (in which $S^k := \mathrm{Sym}^k$)

$$\begin{tikzcd}
C^h \arrow[r, two heads] \arrow[d, two heads] \arrow[drr, dashed, "\xi"] \arrow[dd, dashed, bend left, "\Xi"] & \mathcal{N}\backslash C^h \arrow[r, two heads] \arrow[d, two heads] & \tilde{\mathcal{N}}\backslash C^h = X^h \arrow[d, two heads] \\
\mathcal{S}_h\backslash C = S^h C \arrow[r, two heads] \arrow[d, "AJ_C"] & \mathcal{N}\mathcal{S}_h\backslash C^h \arrow[r, two heads, "\Psi", "m:1 \text{ étale}"'] \arrow[d, phantom, "\shortparallel"] & \tilde{\mathcal{N}}\mathcal{S}_h\backslash C^h = S^h X \arrow[d, two heads, "AJ_X"', "\text{not flat!}"] \\
J(C) \arrow[d, shift left, "P = \sum_{a\in\mathbb{Z}/m}(id - \sigma(a))"] & \mathcal{W} & J(X) \\
A \arrow[u, shift left, "\iota"] & &
\end{tikzcd}$$

$AJ_X : (x_1, \dots, x_h) \mapsto AJ(\sum x_i - hp)$

[AJ maps require a choice of base point on $\underset{p}{X}$ resp. $\underset{\tilde{p}}{\tilde{C}}$]

and consider the preimage of $([K_X] - h\cdot p)$ under AJ_X: this is the linear system $|K_X| \cong \mathbb{P}^{g_X - 1}$. (This fiber of AJ_X is of dimension greater than the relative dim. $= h - g_X = g_X - 2$.) Since $\mathbb{P}^{g_X-1}$ has no nontrivial connected étale cover, $\Psi^{-1}(|K_X|) \cong \coprod^m \mathbb{P}^{g_X - 1}$; denote these components $\{Z_i\}_{i=1}^m$.

The choice of the point $p \in X$ gives an embedding $S^h X \hookrightarrow S^{h+1} X$, and $S^{h+1}X = S^{2g_X - 1} \xrightarrow{AJ_X} J(X)$ is a smooth projective bundle with $|K_X + p| \cong |K_X|$ as fiber. So the normal bundle $N_{|K_X|/S^{h+1}X}$ is trivial, and

just call this $|K_X|$

$$0 \to N_{|K_X|/S^h X} \to N_{|K_X|/S^{h+1}X} \to N_{S^h X/S^{h+1}X}\big|_{|K_X|} \to 0$$

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* Schoen actually proves more, considering non-étale curve coverings. But the étale case are the only one that pertains to any general abelian varieties of Weil type.

gives $c(N_{|K_x|/S^h x}) = c(N_{S^h x/S^{h+1}x}|_{|K_x|})^{-1} = c(\mathcal{O}_{\mathbb{P}^{g_x-1}}(1))^{-1} = (1+H)^{-1}$ hyperplane class

$= 1 - H + H^2 - \cdots \pm H^{g_x-1}$ $\leftarrow h/2$ $\Rightarrow c_{h/2}(N_{|K_x|/S^h x}) \neq 0 \Rightarrow c_{h/2}(N_{Z_1/W}) \neq 0$

$\Rightarrow (Z_1 \cdot Z_1)_W \neq 0$. Define algebraic cycles for each primitive χ

(10)
$$\begin{array}{ccc} Z_\chi := \sum_{a \in \mathbb{Z}/m} \chi(a) Z_a & \in & CH^{h/2}(W)_{\mathbb{Q}(\zeta_m)} \\ \downarrow & & \downarrow \\ \mathfrak{z}_\chi & \in & CH^{h/2}(C^h)^{\mathfrak{S}_h}_{\mathbb{Q}(\zeta_m)} \\ & & \downarrow cl \\ & & H^h(C^h)^{\mathfrak{S}_h}_{\mathbb{Q}(\zeta_m)} \end{array},$$

and note that

(11) $(Z_1 \cdot Z_\chi)_W \neq 0 \Rightarrow cl(Z_\chi) \neq 0 \Rightarrow cl(\mathfrak{z}_\chi) \neq 0$

(12) $\hat{\alpha}^{a*} Z_\chi = \chi(a) Z_\chi$, where $\langle \hat{\alpha} \rangle = \mathrm{Aut}(W/S^h X) \cong \mathbb{Z}/m$.

Under the action of this $\mathbb{Z}/m$, we have the decomposition into <u>1-dim'l</u> eigenspaces

(13) $$U = \bigoplus_\chi H^h(W)^\chi \subset H^h(W) = H^h(C^h)^{\mathfrak{S}_h}$$

which together with (10) & (12) now shows $cl(\mathfrak{z}_\chi) \in U_{\mathbb{Q}(\zeta_m)}$. Since the various Z_χ have "all the eigenvalues χ" in (13) (by (12)), these classes <u>span</u> $U_{\mathbb{Q}(\zeta_m)}$. Taking appropriate $\mathbb{Q}(\zeta_m)$-linear combinations of the $\mathfrak{z}_\chi$ now gives cycles with <u>rational</u> coefficients whose classes span U.

For (a), we remark that $\Xi^* \rho^*$ gives an isomorphism $U' \xrightarrow{\cong} U$. We merely need an inverse operation that also works on the level of cycles. Noting that, up to a mult. constant, $\Xi_* \Xi^*(y) = [\Theta]^{g_C - h} \cup y$ for $y \in H^h(J(C))$, and $[\Theta]^{g_C-h}$ has an algebraic inverse Λ, we have

$$\iota^* \underbrace{\Lambda \Xi_*}(\Xi^* \rho^* v) = \iota^* \rho^* v = v \quad \text{(up to mult. const.)}$$

for $v \in H^h(A)$. Hence $\iota^* \Lambda \Xi_* : U \xrightarrow{\cong} U'$ allows us to "transfer" the cycles from C^h to A. $\square$

But we aren't done, of course. Theorem 1 establishes the Hodge Conjecture for very general "generalized Prym varieties". What about a very general Weil abelian variety?

Our family of abelian varieties A in the above construction lives over a finite cover of the moduli space of genus g_X curves:

$$\begin{array}{ccc} \mathcal{A} & & \\ \downarrow & & \\ \widetilde{\mathcal{M}}_{g_X} & \xrightarrow{\ \ \Phi\ \ } & \Gamma\backslash \mathfrak{h}_{g_X} \end{array}$$

The period mapping for $\mathcal{H}^1(\mathcal{A}/\widetilde{\mathcal{M}}_{g_X})$ factors thru a Shimura variety of PEL type,

$$\mathbb{X} \cong {}_{\Gamma_0}\backslash SU(\tfrac{h}{2},\tfrac{h}{2})^{\times l}/K\ , \quad \left(\begin{array}{l} l = \#\text{ of conj. pairs of complex embeddings of } \mathbb{Q}(\zeta_m) \hookrightarrow \mathbb{C} \\ \quad = \phi(m)/2 \end{array}\right)$$

since the MTG of $V = H^1(A)$ is clearly contained in "$SU(V^{\chi}, h_{\chi})$".

The issue is <u>whether Φ dominates $\mathbb{X}$</u>.

A quick dimension count shows that this can't happen in general:

$\dim \mathbb{X} = \frac{\phi(m)}{2}\left(\frac{h}{2}\right)^2 = \frac{\phi(m)}{2}(g_X - 1)^2$ while $\dim \widetilde{\mathcal{M}}_{g_X} = 3g_X - 3$. The <u>only pairs</u> (g_X, m) for which the second dimension $\geq$ the first are the ones displayed in table (6) [where we equate $m=3$ & $m=6$]. We'll prove the dominance in one case:

<u>Theorem 2</u> (Schoen 1988): There exist families* of Weil abelian 4-folds with $\mathbb{Q}(\zeta_3)$-multiplication, in which the Hodge Conjecture holds for a very general member.

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* by "families" I mean the complete PEL family, and not a smaller dimensional one

(This was improved in his 1998 paper to all families of Weil abelian 4-folds with $\mathbb{Q}(\zeta_3)$-multiplication, and some families of 6-folds.)

Proof: We need to show the map on tangent spaces, $d\Phi$, is surjective. Of course $T_0 M_{g_X} \cong H^1(X, \Theta_X^1)$, while the tangent space to $\mathbb{X}$ is controlled by the movement of the Hodge flag in one piece V^χ of $V = V^\chi \oplus V^{\chi^{-1}}$:

$$\begin{aligned} T_0 \mathbb{X} &\cong \mathrm{Hom}\left(H^{1,0}(C)^\chi, H^{0,1}(C)^\chi\right) \\ &\cong H^{0,1}(C)^\chi \otimes H^{0,1}(C)^{\chi^{-1}} \cong H^1(C,\mathcal{O}_C)^\chi \otimes H^1(C,\mathcal{O}_C)^{\chi^{-1}} \\ &\cong H^1(X, \mathcal{L}) \otimes H^1(X, \mathcal{L}^{-1}) \end{aligned}$$

where $\mathcal{L}$ is a 3-torsion sheaf ($\mathcal{L}^{\otimes 3} = \mathcal{O}_X$) s.t. $\pi_* \mathcal{O}_C \cong \mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^{-1}$.

So

$$d\Phi : H^1(X, \Theta_X^1) \to H^1(X, \mathcal{L}) \otimes H^1(X, \mathcal{L}^{-1})$$

has dual

$$d\Phi^\vee : H^0(X, \mathcal{L} \otimes K_X) \otimes H^0(X, \mathcal{L}^{-1} \otimes K_X) \to H^0(X, K_X^{\otimes 2}),$$

which turns out to be given by multiplication. (But what else could it be?!)

Schoen proves that ~~not~~ every X has a function with divisor of form $3p - 3q$, hence that there exist X for which $|\mathcal{L}^{-1} \otimes K_X|$ is base point free. (See p. 30 of his paper.) Writing $W := H^0(X, \mathcal{L}^{-1} \otimes K_X)$, we therefore have that $W \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{L}^{-1} \otimes K_X$, with kernel $\cong \mathcal{L} \otimes K_X^{-1}$ since $W \cong \Omega^1(C)^{\chi^{-1}}$ has dimension 2. Tensoring by $\mathcal{L} \otimes K_X$ we arrive at the exact sequence

$$0 \to \mathcal{L}^{-1} \to W \otimes \mathcal{L} \otimes K_X \to K_X^{\otimes 2} \to 0;$$

since $H^0(X, \mathcal{L}^{-1}) = \{0\}$, we get that

$$H^0(W \otimes \mathcal{L} \otimes K_X) \cong H^0(X, \mathcal{L} \otimes K_X) \otimes H^0(X, \mathcal{L}^{-1} \otimes K_X)$$

injects into $H^0(X, K_X^{\otimes 2})$. Therefore $d\Phi^\vee$ is injective and $d\Phi$ is surjective, as desired. $\square$