6. Algebraicity of Hodge loci

Recall that given a list of Hodge #'s $\underline{h} = \{h^{p,n-p}\}_{p\in\mathbb{Z}}$, a <u>polarized VHS</u> (of type $\underline{h}$) over a complex analytic manifold is:

$$\text{“}\mathcal{V}\text{''} = (\mathbb{V}, \mathcal{F}^\bullet, Q) = \begin{cases} \mathbb{V} \to \mathcal{S} & \mathbb{Q}\text{-local system} \\ Q : \mathbb{V} \times \mathbb{V} \to \mathbb{Q}_{\mathcal{S}} & (-1)^n\text{-symm. nondegenerate bilinear form} \\ \mathcal{F}^\bullet \subset \mathcal{V}_{\mathcal{O}} := \mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{S}} & \text{decreasing filtration by holo. subbundles} \end{cases}$$

such that the fibers $(\mathbb{V}_s, F^\bullet_s, Q_s)$ are <u>PHS (of type $\underline{h}$)</u> and the flat connection ∇ annihilating $\mathbb{V}$ satisfies the IPR (infinitesimal period rel'n)

$$\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_{\mathcal{S}}.$$

By choosing a reference fiber V_{s_0}, & parallel-translating $\mathcal{F}^\bullet$ to it (from any s), we see this definition is equivalent to having

$$\begin{cases} \left.\begin{array}{l} \mathbb{V} \to \mathcal{S} \\ Q : \mathbb{V} \times \mathbb{V} \to \mathbb{Q}_{\mathcal{S}} \end{array}\right\} \text{ as above} \\ \underline{\Phi : \mathcal{S} \to \Gamma\backslash D_{\underline{h}}} \quad \text{locally liftable, holomorphic, } \Phi^*(\mathcal{I}) = 0 \end{cases}$$

[i.e. $\Phi(\mathcal{S})$ is an integral manifold of $\mathcal{I}$]

where

- Γ is the image of the monodromy representation
  $$\rho : \pi_1(\mathcal{S}) \to G(\mathbb{Q}) \qquad (G = \mathrm{Aut}(V_{s_0}, Q_{s_0})) \quad \text{induced by } \mathbb{V}$$

and

- $\mathcal{I}$ is the differential ideal in $\Omega^\bullet_{D_{\underline{h}}}$ generated by $(TD_{\underline{h}}/W)^*$.

This equivalence of definitions may be refined by replacing the <u>underlined</u> portions by “PHS $\left\{\begin{array}{l}\text{of type } \underline{h} \\ \text{with MTG} \leq \mathcal{M} (\leq G)\end{array}\right.$” and “$\mathcal{S} \to \Gamma\backslash D = \Gamma\backslash \mathcal{M}(\mathbb{R})/H$”.$^{⊛}$

⊛ though one has to be a little careful here (as there is actually a finite collection of D's for $\mathcal{M}$ that one may have to worry about)
$\underset{\widehat{D_{\underline{h}}}}{}$

We say that Φ or $\mathcal{V}$ is (k-)<u>motivic</u> [or <u>of geometric origin</u> (/k)]

if $\exists$ smooth projective morphism $\pi : \mathcal{X} \to \mathcal{S}$ of varieties (defined / $k \subset \mathbb{C}$)

and $a \in \mathbb{N}$ s.t. $\mathcal{V}$ is a sub-VHS of $\mathcal{H}^a(\pi) := (R^a \pi_* \mathbb{Q}, \mathbb{R}^a \pi_* \Omega^{\bullet \geq p}_{\mathcal{X}/\mathcal{S}}$, polarizing form$)$

In the case when $\mathcal{d} \neq 0$ on D, there are uncountably many maximal integral manifolds of $\mathcal{d}$, but only countably many of these can be images of motivic Φ's. (That is, "most" PHS $\in D$ are "not geometric".)

The distribution $\mathcal{W}$ has negative holomorphic sectional curvature, so the "Hodge metric" pulls back to give a negatively curved metric on $\mathcal{S}$. But if $\mathcal{S}$ (say) is a curve that doesn't admit such a metric, like $\mathbb{P}^1 \setminus {}^{\leq 2}$ pts. or an elliptic curve, then Φ is forced to be constant, and $\mathcal{V}$ "isotrivial" (not "constant" b/c $\mathbb{V}$ may still have finite monodromies). Over $\mathbb{P}^1 \setminus 3$ pts., VHS often, but not always, are "hypergeometric" in nature.

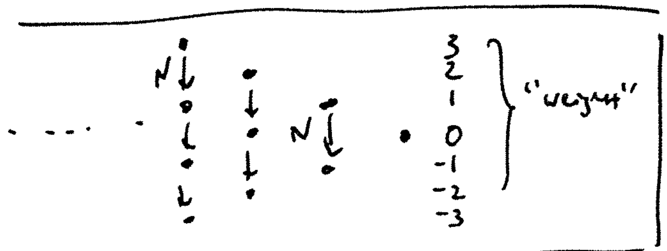
In general the monodromies in a VHS are subject to the following result (again a consequence of the negative curvature):

<u>Monodromy Theorem</u> (Borel, Landman, etc.) : A VHS over Δ^* must have quasi-unipotent monodromy. [That is, if T is the operator $\in Aut(V_{s_0}, G_{s_0})(\mathbb{Q})$ (or $\mathcal{G}(\mathbb{Q})$) that describes the behavior of sections of $\mathbb{V}$ under parallel translation along the counterclockwise loop, then $\exists\, M, N \in \mathbb{Z}$ s.t. $(T^N - id_{V_{s_0}})^M = 0$.]

So T admits a Jordan decomposition $T = T_{ss} T_{un}$ (in $\mathcal{G}(\mathbb{Q})$) s.t. $T_{ss}^N = id$ and $(T_{un} - id)^M = 0$; in particular, the monodromy logarithm $N := \log(T_{un}) = \log(id - (id - T_{un})) = \sum_{j>0} \frac{(-1)^{j-1}}{j} (T_{un} - id)^j$ [finite sum!] is defined. Attached to N is a <u>weight filtration</u> on $\mathbb{V}$: since it is a nilpotent endomorphism $N : V_{s_0} \to V_{s_0}$, $\exists !$ increasing filtration $W(N)_\bullet$ of V_{s_0} s.t.

(1) $\quad N\,W(N)_\ell \subset W(N)_{\ell-2}\,, \qquad N^\ell : Gr_\ell^{W(N)} \xrightarrow{\cong} Gr_{-\ell}^{W(N)} \quad (\forall \ell)$.

In a picture, this looks like



where the strings of dots represent isotypical components for the action of an $sl_2 = \langle N, Y, N^+ \rangle$ that extends the action of N.

We usually shift this "up" so that the weights are centered about the weight ($=n$) of the VHS, viz. $W_\bullet := W(N)_\bullet[-n]$.

The loci of Hodge classes in a VHS are defined as follows: let $v \in W_{s_0}$ be given, assume $\mathcal{V}$ has even weight $2p$, and consider the pullback $\tilde{\mathcal{V}} = (\tilde{W}, \tilde{F}; \tilde{Q})$ to the universal cover $\tilde{S} \xrightarrow{\rho} S$. Writing $\tilde{v} \in \Gamma(\tilde{S}, \tilde{W})$ for the section extending v, we set $\tilde{S}(v) := \{\tilde{s} \in \tilde{S} \mid \tilde{F}^p_{\tilde{s}} \ni \tilde{v}\}$ and

$$S(v) := \rho(\tilde{S}(v)),$$

the Hodge locus of v. The question "is $s_0 \in S(v)$?" is more complicated than it may seem: it really means,

$$\text{is } F^p_{s_0} \cap \Gamma \cdot v \text{ nonempty?}$$

(Also note that we could have $S(v) = S$ or $\emptyset$.)

First we should point out why, at least, $S(v)$ is an analytic subvariety of S (cut out locally by holomorphic functions). Say $s_0 \in S(v)$, and let $\mathcal{B} \ni s_0$ be a (relatively compact) ball. Then for each $\gamma \in \Gamma$, holomorphicity of F^p implies that

$$\Sigma(\gamma v) := \{ s \in \mathcal{B} \mid \gamma v \in F^p \}$$

is analytic; the problem is that

(2) $$\mathcal{S}(v) \cap \mathcal{B} = \bigcup_{\gamma \in \Gamma} \Sigma(\gamma v)$$

could be an infinite union (which we must rule out). Writing $\langle u, v \rangle_s := Q(\varphi_s(i) u, \bar{v})$, $\quad s \in \Sigma(\gamma v) \implies$

(3) $$\langle \gamma v, \gamma v \rangle_s \underset{\uparrow}{=} Q(\gamma v, \gamma v) = Q(v, v).$$

($\varphi_s(i)$ acts on type (p,p) class by $i^{p-p} = 1$, and $\overline{\gamma v} = \gamma v$ (because rational))

Now Γ stabilizes a $\mathbb{Z}$-local system $\mathbb{V}_{\mathbb{Z}} \subset \mathbb{V}$, and (as we may assume that v is integral by taking a multiple) the elements γv lie in $V_{\mathbb{Z}} := \mathbb{V}_{\mathbb{Z}}|_{s_0}$. Moreover, the $\{\langle\, , \rangle_s\}_{s \in B}$ are uniformly equivalent to $\langle\, , \rangle_{s_0}$; since this is definite, only finitely many (integral!) γv can satisfy (3) for some $s \in B$. Hence, all but finitely many of the $\Sigma(\gamma v)$ are empty, the union (2) is finite, and $\mathcal{S}(v) \cap \mathcal{B}$ is analytic.

On the other hand, if we assume $\mathcal{S}$ is algebraic, and Φ is k-motivic, the Hodge conjecture would imply much more:

<u>Proposition 1</u>: If the HC holds,

(a) $\mathcal{S}(v)$ is algebraic

(b) $\mathcal{S}(v)$ is defined $/\bar{k}$.

<u>Proof (Sketch)</u>: If a component V of $\mathcal{S}(v)$ is non-algebraic of dimension δ, then it has a point $p =: s_0$ of $\bar{k}$-transcendence degree $> \delta$. Let $Z \subset X_{s_0}$ be the cycle with class v, and first note that we may assume

Z is defined over $\bar{k}(s_0)$ (by specializing). Next, take the $\bar{k}$-spread $\mathcal{Z}$ of Z over the $\bar{k}$-spread $\mathcal{S}_0 \subset \mathcal{S}$ of s_0 (with fiber Z over s_0). The cycle classes of the fibers of $\mathcal{Z} \to \mathcal{S}_0$ are the parallel translates of $v = [Z]$, and they are Hodge classes. This contradicts $\dim V = \delta$.

If V is assumed algebraic, the same spread argument with s_0 = very general point in V gives that $\mathcal{S}_0 \supset V$, with equality iff V def'd $/\bar{k}$ (and $\supsetneq$ is again a contradiction). $\square$

We'll return to (b) in a later section.

Given a k-motivic $\mathcal{V} \to \mathcal{S}$ (or Φ), the $T^{a,b}V$ (tensor spaces) are also PVHS, and we can consider <u>their</u> loci of Hodge classes. This recovers the $\mathcal{S}(t)$ of §I.B.4, and $\bigcup_{t:\mathcal{S}(t)\neq\mathcal{S}} \mathcal{S}(t)$ the "Mumford-Tate Noether-Lefschetz locus" where the MT group drops from its generic one. In fact, this is nothing but the Φ-preimage of all (Γ-quotients of) MT-subdomains in D.

In general, if we have an ambient MT or period domain D, with ambient MTG $\mathcal{G}$, and we look at the sublocus $D' \subset D$ consisting of Hodge structures with MTG $\leq M$ (for some appropriate reductive $\mathbb{Q}$-sgp. $M \leq \mathcal{G}$), this D' will consist of finitely many connected components D_i, each of which is an $M(\mathbb{R})$-orbit (i.e. MT subdomain). Then Proposition 1 has the immediate

<u>Corollary</u>: (of (a)) $\Phi^{-1}(\Gamma_i \backslash D_i)$ is algebraic $\subset \mathcal{S}$
(of (b)) $\Phi^{-1}(\Gamma_i \backslash D_i)$ is defined $/\bar{k}$.
} (for Φ $\bar{k}$-motivic, & assuming the Hodge conj.)

The rest of this section is devoted to sketching the proof of

<u>Theorem 1 (Cattani-Deligne-Kaplan)</u>: (a) in Prop. 1 is true without assuming the Hodge Conjecture and without assuming Φ is motivic. That is, the components of the Hodge locus $\bigcup_{v:\mathcal{S}(v)\neq\mathcal{S}} \mathcal{S}(v)$ of an arbitrary polarized VHS over $\mathcal{S}$ are algebraic, under the sole assumption that $\mathcal{S}$ is algebraic!

<u>Remarks</u>: (i) Note that since a closed analytic subvariety of a projective algebraic variety is algebraic (Chow's Thm.), we basically need to show that the $\overline{\mathcal{S}(v)}$ don't behave wildly on $\bar{\mathcal{S}}\setminus\mathcal{S}$ (the bar denotes analytic closure of $\mathcal{S}(v)$ in a compactification $\bar{\mathcal{S}}\supset\mathcal{S}$).

(ii) This is sometimes billed as evidence for the Hodge conjecture, but this requires some faith that the plethora of "non-motivic" integral manifolds of $\mathcal{d}$ somehow gets largely eliminated by having to be closed under the action of Γ.

In the discussion that follows, we shall assume $\mathcal{S}$ is algebraic (quasi-projective), and write $D = G(\mathbb{R})/\mathcal{H}$, where $G = Aut(V, Q)$ is an orthogonal group since V is of even weight.

Let $\bar{\mathcal{S}}\supset\mathcal{S}$ be a good compactification (i.e. $\bar{\mathcal{S}}$ smooth projective, $\bar{\mathcal{S}}\setminus\mathcal{S}$ a NCD = normal crossing divisor). A neighborhood of a point $\underline{0}\in\bar{\mathcal{S}}$ looks like $\bar{\mathcal{N}}\subseteq\Delta^{r+l}$, where $\bar{\mathcal{N}}\cap\mathcal{S} = (\Delta^*)^r\times\Delta^l =: \mathcal{N}$.

(By taking slices and pulling back to finite covers, we may assume that $l=0$ and $\pi_1((\Delta^*)^r)$ acts unipotently on the VHS's underlying local system $\mathbb{V}$.)

We need to show that for each neighborhood, the analytic closure of $\mathcal{S}(v) \cap \mathcal{N}$ in $\bar{\mathcal{N}}$ is still an analytic subvariety (cut out by holo. equations). Then

$$\overline{\mathcal{S}(v)} \subset \bar{\mathcal{S}} \text{ analytic} \underset{\text{GAGA/Chow's thm.}}{\Longrightarrow} \overline{\mathcal{S}(v)} \text{ algebraic} \Rightarrow \mathcal{S}(v) \text{ algebraic}.$$

So Theorem 1 reduces to

Proposition 2: Given a $\mathbb{Z}$-VHS[⊛] $\mathcal{V} = (\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}, Q)$ over $\underset{(s_0)}{\mathcal{N} := (\Delta^*)^r}$, and $\alpha \in \mathbb{N}$, there exist

$$v_1, \ldots, v_N \in V_{\mathbb{Z}, s_0}$$

such that (after possibly shrinking $\bar{\mathcal{N}}$)

(i) for any $s \in \mathcal{N}$ and $u \in V_{\mathbb{Z},s} \cap V_s^{p,p}$ with $Q(u,u) = \alpha$,

u is the parallel translate of one of the $\{v_j\}_{j=1}^N$ along a short path $\overrightarrow{s.s_0}$;

(ii) denoting by v_j as well the extension of $v_j \in V_{\mathbb{Z},s_0}$ to a multivalued flat section$/_{\mathcal{N}}$ of $\mathbb{V}_{\mathbb{Z}}$, $\overline{\mathcal{N}(v_j)} \subset \bar{\mathcal{N}}$ is an analytic subvariety (for each j); and

(ii) v_j is invariant under $\pi_1(\mathcal{N}(v_j))$.

The one apparent issue about passing from Prop. 2 to Thm. 1 is this: we can't locally discern what all the γv_j look like, for $\gamma \in \Gamma$ (global monodromy group over $\mathcal{S}$). But this doesn't matter; they all have the same Q-norm α, hence by (i) only finitely many of them contribute!

We now examine the period map in this context:

⊛ that is, we refine $(\mathbb{V}, \mathcal{F}^{\bullet}, Q)$ to $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}, Q)$, where $Q : \mathbb{V}_{\mathbb{Z}} \times \mathbb{V}_{\mathbb{Z}} \to \mathbb{Z}_{\mathcal{S}}$

we have

$$\begin{tikzcd} \underline{z} \in \mathfrak{h}^r \arrow[r, "\tilde{\Phi}"] \arrow[d, "\rho"] & D \arrow[d] \\ (\Delta^*)^r \arrow[r, "\Phi"] & \Gamma\backslash D \end{tikzcd}$$

$\underline{z} \mapsto (e^{2\pi i \underline{z}} =) \underline{s}$

with $\tilde{\Phi}$ holomorphic & horizontal, satisfying $\tilde{\Phi}(\underline{z}+e_j) = T_j^{-1}\tilde{\Phi}(\underline{z})$.

(Since $\pi_1((\Delta^*)^r) \cong \mathbb{Z}^r$, the $\{T_j\}$ commute.) Write $R_\beta := \{a+bi \mid b>\beta,\ |a|\leq 1\}$.

Asymptotic Hodge theory (remarks) [here $V = \mathbb{Q}$-vector space]

(A) Schmid's "nilpotent orbit theorem": $\Psi(\underline{s}) := e^{\sum z_j N_j}\tilde{\Phi}(\underline{z})$ extends across $\underline{0}$ as a holomorphic map $\Psi : \Delta^r \to \check{D}$, and the nilpotent orbit $\tilde{\Theta}(\underline{z}) := e^{-\sum z_j N_j}\Psi(\underline{0})$ descends to $\Theta : \underset{\text{(poss. shrinking)}}{(\Delta^*)^r} \to \Gamma\backslash D$. The flag $\Psi(\underline{0}) \in \check{D}$ is called the limiting flag $F^\bullet_{\lim}$.

(B) Limiting mixed Hodge structure (LMHS): let $W_\bullet := W(N_1+\cdots+N_r)[-2p]$ be the weight monodromy filtration. For a flag $F^\bullet$ on V (i.e. $\in \check{D}$), $(V, F^\bullet, W_\bullet)$ is a mixed Hodge structure $\overset{\text{def.}}{\iff}$ $(Gr_i^W V, F^\bullet|_{Gr_i^W})$ is a HS of weight i (for each i).
This MHS is polarized by {a nilpotent endomorphism $N' \in End(V)$ / and a bilinear form $Q : V\times V \to \mathbb{Q}$} $\overset{\text{def}}{\iff}$ $N'F^\bullet \subset F^{\bullet-1}$, $W(N')_\bullet[-2p] = W_\bullet$, and the HS induced by $F^\bullet$ on $\ker\{(N')^{j+1} : Gr^W_{2p+j} \to Gr^W_{2p-j-2}\}$ is polarized by $Q(\cdot, (N')^j\cdot)$ for each $j \geq 0$. (*)
Schmid's "SL_2-orbit theorem" implies that $(V, F^\bullet_{\lim}, W_\bullet)$ ($W_\bullet$ as above) is a MHS (called the LMHS), polarized by every $\sum \lambda_i N_i$, $\lambda_j \in \mathbb{R}_+$.

(C) Deligne bigrading of a MHS: given $(V, F^\bullet, W_\bullet) = $ MHS, there is a unique decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} I^{p,q}(V)$$

s.t. $F^a V_{\mathbb{C}} = \bigoplus_{p,q:\ p\geq a} I^{p,q}$, $W_b V_{\mathbb{C}} = \bigoplus_{p,q:\ p+q\leq b} I^{p,q}$, and $\overline{I^{q,p}} \equiv I^{p,q} \mod \bigoplus_{a,b:\ a<p,\, b<q} I^{a,b}$.

We say $(V, F^\bullet, W_\bullet)$ is $\mathbb{R}$-split $\iff$ $\overline{I^{q,p}} = I^{p,q}$ $(\forall p,q)$. //

(*) here Q is the "ambient polarization" (implicit in $\check{D}$) of the original PVHS.

Proposition 2(i) now takes the form (writing $V_{\mathbb{Z}} = \mathbb{V}_{\mathbb{Z},s_0}$, and v for a multivalued locally flat section of $\mathbb{V}$ and the corr. element of $V_{\mathbb{Z}}$)

Proposition 3: (a) For any $K \in \mathbb{R}_+$,

$$V_\beta := \left\{ v \in V_{\mathbb{Z}} \;\middle|\; \begin{array}{l} \bullet\ Q(v,v) \leq K \\ \bullet\ v \in F^{p}_{\tilde{\Phi}(\underline{z})},\ \text{for some } \underline{z} \in (\mathcal{R}_\beta)^{\times r} \end{array} \right\}$$

is finite for $\beta \gg 0$.

(b) If $\overline{\mathcal{N}(v)} \ni \underline{0}$, then $v \in F^p_{\lim} \cap \overline{F^p_{\lim}} \cap W_{2p} \subset I^{p,p}(V)$.

We won't completely prove this finiteness statement, and for now we postpone it, turning to Prop. 2(ii) and (iii) (for which we may use Prop. 3).

Fix $v \in V_{\mathbb{Z}}$ s.t. $Q(v,v) \leq K$ & $\overline{\mathcal{N}(v)} \ni \underline{0}$. The bigrading of $V_{\mathbb{C}}$ arising from $(F^{\bullet}_{\lim}, W_\bullet)$ gives a bigrading $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p,q} \mathfrak{g}^{p,q}$, with $\mathfrak{g}^{p,q}$ sending $I^{a,b}(V) \to I^{a+p,b+q}(V)$, and hence $\bigoplus_{\substack{p,q:\\ p\geq 0}} \mathfrak{g}^{p,q}$ stabilizing the flag $F^{\bullet}_{\lim}$. Let $\mathfrak{b} := \bigoplus_{\substack{p,q\\ p<0}} \mathfrak{g}^{p,q}$ be the complement; this is the tangent space to $\check{D}$ at $\Psi(\underline{0})$, and so there exists a holomorphic function

4) $$(\cdots + \Gamma^{-2} + \Gamma^{-1} =)\ \Gamma : \Delta^r \longrightarrow \mathfrak{b}\ (= \mathfrak{b}^{-1} \oplus \mathfrak{b}^{-2} \oplus \cdots)$$

such that

5) $$\Psi(\underline{s}) = e^{\Gamma(\underline{s})} F^{\bullet}_{\lim}.$$

Defining a holo. fcn.

6) $$(\cdots + X^{-2} + X^{-1} =)\ X : \mathfrak{h}^r \longrightarrow \mathfrak{b}\ (= \mathfrak{b}^{-1} \oplus \cdots)$$

by

7) $$e^{X(\underline{z})} := e^{-\sum z_i N_i} e^{\Gamma(\underline{s})},$$

we have

8) $$\tilde{\Phi}(\underline{z}) = e^{X(\underline{z})} F^{\bullet}_{\lim}$$

which is called the <u>local normal form</u> of the VHS $\mathbb{V}$.

Now we get to see the power of the horizontality condition on $\tilde{\Phi}$, τ
(i.e. Griffiths transversality)

which $\Longrightarrow \quad d e^{X} F_{\lim} \subset e^{X} F_{\lim}^{\bullet-1} \otimes \Omega^1_{h^r}$

$\Longrightarrow \quad e^{-X} d e^{X} \in \Big\{ \Big(\bigoplus_{\substack{p,q:\\ p\geq -1}} \mathfrak{g}^{p,q} \Big) \cap \underset{\text{by (6)}}{\mathfrak{b}} \Big\} \otimes \Omega^1_{h^r} = \mathfrak{b}^{-1} \otimes \Omega^1_{h^r}$

$\Longrightarrow \quad e^{-X} d e^{X} = d e^{X^{-1}}$ using: $e^{-X}de^{X} = e^{-\Gamma(s)} e^{\sum z_i N_i} d\big(e^{-\sum z_i N_i} e^{\Gamma(s)}\big)$
$= -\sum_j e^{-\operatorname{ad}\Gamma} N_j \otimes dz_j + e^{-\Gamma} d e^{\Gamma}$

$\Longrightarrow \quad -e^{-\operatorname{ad}\Gamma} N_j + 2\pi i s_j e^{-\Gamma} \frac{\partial}{\partial s_j} e^{\Gamma} \in \mathfrak{b}^{-1}$ using: $e^{-\operatorname{ad}\Gamma} N_j \equiv N_j \mod \mathfrak{b}^{-2}+\mathfrak{b}^{-3}+\cdots$

$\underset{s_j=0}{\Longrightarrow} \quad N_j - [\Gamma, N_j] + \frac{1}{2}[\Gamma,[\Gamma,N_j]] - \cdots \in \mathfrak{b}^{-1}$

$\Longrightarrow \quad (\operatorname{ad} N_j)\,\Gamma(s) = 0$ at $s_j = 0$.

Now

$$\begin{aligned}
\widetilde{N(v)} &:= \{ \underline{z} \in h^r \mid v \in F^p_{\Phi(\underline{z})} \} \\
&\underset{(7)}{=} \{ \;''\; \mid e^{-X(\underline{z})} \cdot v \in F^p_{\lim} \} \\
&= \{ \;''\; \mid e^{-X(\underline{z})} v = v \} \\
&= \{ \;''\; \mid X(\underline{z}) v = 0 \} \\
&= \{ \;''\; \mid X^{-j}(\underline{z}) v = 0 \;\; (\forall j) \} \\
&\subset \{ \;''\; \mid X^{-1}(\underline{z}) v = 0 \} =: \widetilde{N(v)}_{-1},
\end{aligned}$$

using: $\begin{cases} v \in I^{p,p} & \text{by Prop. 3(b), and} \\ e^{X(\underline{z})} v \equiv v \mod \bigoplus_{\substack{a,b\\ a<p}} I^{a,b} \end{cases}$

and we have a diagram (defining $N(v)_{-1}$)

$$\begin{tikzcd}
\widetilde{N(v)} \arrow[d, two heads] & \subset & \widetilde{N(v)}_{-1} \arrow[d, two heads] & \subset & h^r \arrow[d, two heads, "\rho"] \\
N(v) & \subset & N(v)_{-1} & \subset & (\Delta^*)^r
\end{tikzcd}$$

Claim 1: $\overline{N(v)_{-1}} \subset \Delta^r$ is analytic.

Pf: $0 = X^{-1} v = -\sum z_i N_i v + \Gamma^{-1} v \Longrightarrow$ (up to change of basis in the z_i / s_i)

$\overline{N(v)_{-1}}$ is given by equations $\begin{cases} z_j + \mu_j(\underline{s}) = 0, & j \leq j_0 = \dim(\operatorname{span}\{N_i v\}) \\ \mu_j(\underline{s}) = 0, & j > j_0 \end{cases}$

$= \begin{cases} s_j = e^{-2\pi i \mu_j(\underline{s})}, & j \leq j_0 \\ \mu_j(\underline{s}) = 0, & j > j_0 \end{cases}$ with the $\mu_j : \Delta^r \to \mathbb{C}$ holomorphic.

$\square$

Claim 2 : v is $\pi_1(\mathcal{N}(v)_{-1})$ - invariant.

Pf : Let $\varsigma : [0,1] \to \widetilde{\mathcal{N}(v)_{-1}} + \mathbb{Z}^r$, $0 \mapsto \underline{z}_0$. have $\underline{m} := \varsigma(1) - \varsigma(0) \in \mathbb{Z}^m$, so that $\sigma = p \circ \varsigma$ is a loop in $\mathcal{N}(v)_{-1}$ with monodromy image $T := \prod_j T_j^{m_j}$.

Now $\varsigma(t) = \tilde{\varsigma}(t) + \underline{m}'$ for some $\tilde{\varsigma} : [0,1] \to \widetilde{\mathcal{N}(v)_{-1}}$, $\underline{m}' \in \mathbb{Z}^m$; since Γ^{-1} depends only on $\underline{s}$, we have (using $X^{-1} = -\sum z_i N_i + \Gamma^{-1}$)

$$X^{-1}(\varsigma(t)) = X^{-1}(\tilde{\varsigma}(t)) + \sum m_j' N_j$$
$$= \sum m_j' N_j \qquad (X^{-1}(\tilde{\varsigma}(t)) = 0 \text{ by defn. of } \widetilde{\mathcal{N}(v)_{-1}})$$

$\Rightarrow X^{-1}(\varsigma(t))v = \sum m_j' N_j v$ is constant

$\Rightarrow -\sum m_j N_j v = X^{-1}(\varsigma(1))v - X^{-1}(\varsigma(0))v = 0$

$\Rightarrow Tv = v$. (since $T = e^{\sum m_j N_j}$) $\square$

Claim 3 : $\overline{\mathcal{N}(v)}$ is a union of irreducible components of $\overline{\mathcal{N}(v)_{-1}}$, hence is analytic.

Pf : $\mathbb{V} \to (\Delta^*)^r$ unipotent $\Rightarrow \exists$ canonical extension $\mathcal{V}_e \to \Delta^r$.

[In more detail : since $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_{\mathcal{N}}$, we can trivialize $\mathcal{V}$ by writing it as $\underbrace{e^{-\sum z_i N_i}\mathbb{V}}_{\text{single-valued!}} \otimes \mathcal{O}_{\mathcal{N}}$, then extend to $\overline{\mathcal{N}}$.

The nilpotent orbit theorem above $\Rightarrow$ the $\mathcal{F}^i$ extend as holo. subbundles $\mathcal{F}_e^i \subset \mathcal{V}_e$.]

In particular,

$$\begin{array}{ccc} \underline{z} & \longmapsto & e^{X(\underline{z})}v = e^{-\sum z_i N_i} e^{\Gamma(s)} v \\ \downarrow & & \downarrow \\ \underline{s} & \longmapsto & \hat{v}(s) \in \mathcal{V}_e \end{array}$$

descends to $\underline{s} \mapsto \hat{v}(s) \in \mathcal{V}_e$

and $\mathcal{N}(v) = \{ \underline{s} \in (\Delta^*)^r \mid X(\underline{z})v = 0 \text{ for some } \underline{z} \in p^{-1}(\underline{s}) \}$

$= \{ \quad '' \quad \mid \hat{v}(s) = Tv \text{ for some } T = \prod T_i^{n_i} \}$

But then

$$\left\{\begin{array}{l} e^{-X} d e^{X} = d e^{X^{-1}} \\ X^{-1}(\underline{z}) \cdot v = 0 \quad \text{for } \underline{z} \in \overline{\mathcal{N}(v)_{-1}} \end{array}\right. \implies e^{-X} d\hat{v} = d e^{X^{-1}} v = 0$$

$\implies$ $\hat{v}(s)$ defines on $\mathcal{N}(v)_{-1}^{(sm)}$ ← smooth locus
a flat holo. section of $\mathcal{V}$.

So if $\mathcal{Y} \subset \overline{\mathcal{N}(v)_{-1}}$ is an irred. component (so that $\mathcal{Y}^{sm}$ & $\mathcal{Y}^{sm} \cap \Delta^{*r}$ are connected), then at <u>any</u> $\underline{s}_0 \in \mathcal{Y}^{sm} \cap \mathcal{N}(v)$, $\hat{v}(s_0) = T_0 v$. But since $\hat{v}$ is flat on $\mathcal{Y}$, $\hat{v}(\underline{s}_0) = T_0 v$ $\forall \underline{s} \in \mathcal{Y}^{sm} \cap \mathcal{N}(v)$. Hence $\mathcal{Y} \cap \overline{\mathcal{N}(v)} = \phi$ or $\mathcal{Y} \subset \overline{\mathcal{N}(v)}$. $\square$

So Claims 3 & 1 $\Rightarrow$ (ii) in Prop 2 (for each v_j), and
Claims 3 & 2 $\Rightarrow$ (iii) in Prop. 2;

we have reduced the proof of the CDK theorem to Prop. 3 (the finiteness statement), which we shall now explain <u>in the 1-variable case</u>.

Henceforth, for notational simplicity, assume "$p=0$", i.e. twist ($\mathbb{Q}(p)$ = trivial HS of weight $-2p$) the VHS Φ so that it is in weight 0 rather than $2p$. This helps so that $W_\bullet = W(N)_\bullet$ (w/o $[-2p]$ shift) and the weights are centered about zero. We also reverse the sign of N (to eliminate lots of signs). A few more preliminaries are needed:

<u>More on asymptotics</u>

(D) <u>Deligne's splitting</u>: let $(V, F^\bullet, W_\bullet)$ be a MHS polarized by (N, Q) in the sense of "(B)" above (p. 8). Then (following "(C)") set $W_\bullet = W(N)_\bullet$ and $V_{\mathbb{C}} = \oplus I^{a,b}_{(F,W)}$, and (as on p. 9) write $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{a,b} \mathfrak{g}^{a,b}_{(F,W)} \supset \Big(\bigoplus_{\substack{a<0 \\ b<0}} \mathfrak{g}^{a,b}_{(F,W)} \Big) \cap \mathfrak{g}_{\mathbb{R}} =: \Lambda_{(F,W)}$. We state w/o proof

<u>Lemma 1 (Deligne)</u>: (a) $\exists !\ \delta \in \Lambda_{(F,W)}$ s.t. $(V, F_\delta := e^{-i\delta} F^\bullet, W_\bullet)$ is $\mathbb{R}$-split.

(b) $\Lambda_{(F_\delta, W)} = \Lambda_{(F,W)}$ and $[\delta, N] = 0$.

Define semisimple $Y_\delta \in \mathrm{End}_{\mathbb{R}MHS}(V, F_\delta^\bullet, W_\bullet)$ to multiply $I^{a,b}_{(F_\delta,W)}$ by $a+b$; the eigenspaces are $E_j(Y_\delta) = \bigoplus_{a+b=j} I^{a,b}_{(F_\delta,W)}$.

<u>Exercise 1</u>: (i) $N \in \mathfrak{g}_{\mathbb{Q}} \cap \mathfrak{g}^{(-1,-1)}_{(F,W)} = \mathfrak{g}_{\mathbb{Q}} \cap \mathfrak{g}^{(-1,-1)}_{(F_\delta,W)}$

(ii) $Y_\delta \in \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}^{(0,0)}_{(F_\delta,W)}$

<u>Lemma 2</u>: $F^0 \cap W_0 \cap V_{\mathbb{R}} \subset E_0(Y_\delta) \cap \ker(\delta)$

<u>Pf</u>: $v \in \mathrm{LHS} \subset I^{0,0}_{(F,W)} = e^{i\delta} I^{0,0}_{(F_\delta,W)} \Rightarrow e^{i\delta} f = v = \bar{v} = e^{-i\delta} \bar{f}$

$\underset{(F_\delta,W)\ \mathbb{R}\text{-split}}{\Rightarrow} e^{2i\delta} f = \bar{f} \in I^{0,0}_{(F_\delta,W)}$

$\underset{\delta \in \Lambda}{\Rightarrow} \delta f = 0$ and $f = \bar{f} = v$. $\square$

(E) <u>An asymptotic calculation</u>: Suppose given a unipotent VHS $\mathcal{V} \to \Delta^*$ of weight 0 with associated period map

$$\tilde{\Phi}(z) = e^{zN} e^{\Gamma(s)} F^\bullet : \mathfrak{h} \to D,$$

so that $(F^\bullet, W_\bullet)$ is its LMHS, and write F_δ, Y_δ, and

$$e(y) := \exp\{\tfrac{1}{2}(\log y)\, Y_\delta\} \in G(\mathbb{R}) \qquad (y \in \mathbb{R}_+).$$

<u>Exercise 2</u>: (i) $e(y) F_\delta^\bullet = F_\delta^\bullet$

(ii) $e(y)\, e^{\mu}\, e(y)^{-1} = e^{y^{a+b}\mu}$ for $\mu \in I^{a,b}_{(F_\delta,W)}$.

<u>Lemma 3</u>: writing $z = x+iy$, $|x|$ bounded, we have

$$\lim_{y \to \infty} e(y) \tilde{\Phi}(z) = F_\#^\bullet := e^{iN} F_\delta^\bullet.$$

Pf: $\tilde{\Phi} = e^{xN} e^{iyN} e^{\Gamma} F^{\cdot} = e^{xN} e(y)^{-1} e^{iN} e(y)\, e^{\Gamma} e^{i\delta} F_{\delta}^{\cdot}$

$$\Rightarrow e(y)\tilde{\Phi} = e^{\frac{x}{y}N} e^{iN} \underbrace{\left(e(y) e^{\Gamma} e(y)^{-1}\right)}_{\to 1} \underbrace{\left(e(y) e^{i\delta} e(y)^{-1}\right)}_{\to 1} F_{\delta}^{\cdot} \xrightarrow[y\to\infty]{} \underline{\underline{e^{iN} F_{\delta}^{\cdot}}}.$$

(use the exercise) (Γ has some positive Y_δ-eigenvalues, but $|\Gamma(s)| \sim |s| = e^{-2\pi y}$ and $y^j e^{-2\pi y} \to 0$) (δ has negative Y_δ-eigenvalues)

□

Lemma 4: $F_{\#}^{0} \cap W_0 \cap V_{\mathbb{R}} \subset E_0(Y_\delta) \cap \ker(N)$.

Pf: Replace $F^{\cdot}$ by $F_{\#}^{\cdot}$ in Lemma 2, noting that the Deligne splitting in this case is $e^{-iN} F_{\#}^{\cdot} = e^{-iN} e^{iN} F_{\delta}^{\cdot} = F_{\delta}^{\cdot}$ □

Since $e(y) \in G(\mathbb{R})$ and $\tilde{\Phi}(z) \in D$, Lemma 3 shows that $F_{\#}^{\cdot} \in D$. //

Now write $\|u\|_{\tilde{\Phi}(z)} := Q\left(\varphi_{\tilde{\Phi}(z)}(i)\, u, \bar{u}\right)^{1/2}$ $(\in \mathbb{R}_{>0})$ for the Hodge norm. (No subscript means $F_{\#}^{\cdot}$.) Proposition 3 in the 1-variable case says that (writing $V_{\mathbb{Z}}$ for the reference fiber of $\mathcal{V}$ over $s_0 \in \Delta^*$)

(a) $V_\beta(K) := \{ v \in V_{\mathbb{Z}} \mid Q(v,v) \le K \text{ and } v \in F^{0}_{\tilde{\Phi}(z)} \text{ for some } z \in \mathcal{R}_\beta \}$ is finite for $\beta \gg 0$

(b) $\mathcal{N}(v) \ni \underline{0}$ (for some $v \in V_{\mathbb{Z}}$) $\Longrightarrow$ $v \in F^{0}_{\lim} (\cap \overline{F^{0}_{\lim}}) \cap W_0$.

Fixing a $K \in \mathbb{R}_{+}$, this reduces to

Proposition 4: Let $u_n \in V_{\mathbb{Z}}$, $z_n = x_n + i y_n \in \mathfrak{h}$ be sequences such that

(i) $Q(u_n, u_n) \le K$

(ii) $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} y_n = \infty$

(iii) $u_n \in F^{0}_{\tilde{\Phi}(z_n)} =: F^{0}_{n}$ $(\forall n)$

Then we have

(iv) for $n \gg 0$, u_n takes values in a finite set $\underbrace{\{v_1, \ldots, v_n\}}_{\Uparrow\,(a)} \underbrace{\subset F^{0}_{\lim} \cap W_0}_{\Uparrow\,(b)}$.

Proof of Prop. 4:* Set $e_n := e(y_n)$.

Step 1 : $\|e_n u_n\|$ is bounded.

$u_n \in F^0_{\tilde{\Phi}(z_n)} \Rightarrow Q(u_n, \bar{u}_n) = \|u_n\|^2_{\tilde{\Phi}(z_n)}$

$e_n \in \mathrm{Aut}(V_{\mathbb{R}}, Q) \Rightarrow \qquad = \|e_n u_n\|^2_{e_n\tilde{\Phi}(z_n)}$

$e_n\tilde{\Phi}(z_n) \to F^{\cdot}_{\#}$ (Lemma 3) $\Rightarrow \qquad \geq c_0 \|e_n u_n\|^2$ (because the norms are uniformly bounded in a compact set)**

basic calculation: for $g \in \mathrm{Aut}(V_{\mathbb{R}}, Q)$, $\varphi \leftrightarrow F \in D$, $Q(\varphi(i)u, \bar{u}) = Q(g\varphi g^{-1}(i)gu, \overline{gu}) \Rightarrow \|u\|_F = \|gu\|_{gF}$.

Now the Claim follows from hypothesis (i).

Step 2 : $u_n \in W_0$ and "$Gr^W_0 u_n$" $\subset$ finite set

Write $u_n = \sum u_n^\lambda$, $u_n^\lambda \in E_\lambda(Y_\sigma)$, $\lambda_0 :=$ largest λ s.t. "$u_n^\lambda \neq 0$ for infinitely many n"

$\xrightarrow[\text{(Exercise 2)}]{} \|e_n u_n\| \sim y_n^{\lambda_0/2}\|u_n^{\lambda_0}\|$

$\xrightarrow[\text{(Step 1)}]{}$ $\begin{cases} \lambda_0 \leq 0 \Longrightarrow u_n \in W_0 \ (\forall n \gg 0) \\ \text{AND} \\ \|u_n^0\| \text{ bounded } (\& \ u_n^0 \in Gr^W_0 V_{\mathbb{Z}}) \Longrightarrow u_n^0 \text{ vary in finite set } \{v_1, \ldots, v_m\} \subset E_0(Y_\sigma) \end{cases}$

[poss. incl. 0]

Break $\{u_n\}$ accordingly into m subsequences, and consider just one $\rightsquigarrow$ $m = 1$, $v = v_1$, $u_n = v + w_n$, $w_n \in W_{-1}$ and $v \in E_0(Y_\sigma)$.

Step 3 : $u_n \in \ker N$

$u_n \in F^0_n \cap W_0 \cap V_{\mathbb{Z}}$ and $e_n W_0 = W_0$

** Important point here: $F^{\cdot}_{\#} \in D$. We can't do this with $F_{\lim}$ or $F^{\cdot}_\sigma$, because $F_{\lim}$ may not be in D and $F^{\cdot}_\sigma$ never is.

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* This will establish the full result of [CDK] for 1-variable VHS; the several-variable version requires the very technical results and notations of Cattani-Kaplan-Schmid, so I won't cover this.

$$\Longrightarrow \quad [e_n u_n] \longrightarrow \text{line } \ell \subset F^0_\# \cap W_0 \cap V_{\mathbb{R}} \subset E_0(Y_\sigma) \cap \ker(N)$$

Lemmas 3 & 4 $\left(\begin{array}{c} \cap \\ \mathbb{P} V_{\mathbb{R}} \end{array} \right)$

$$\Longrightarrow \quad \ell \subset F^0_\sigma \cap W_0 \cap V_{\mathbb{R}} \subset I^{0,0}_{(F_\sigma, W)} =: I^{0,0}_\sigma$$

$\left(F^0_\sigma = e^{-iN} F^0_\# \text{ \& } \ell \subset \ker(N) \right)$

But ℓ also (as the limit of $[e_n u_n] = [v + e_n w_n]$) clearly projects to $[v]$ under

$$\begin{array}{ccc} V & \twoheadrightarrow & E_0(Y_\sigma) \\ \cup & \nearrow & \\ I^{0,0}_\sigma & & \end{array}$$

$$\Longrightarrow \begin{cases} \ell = [v] \\ e_n w_n \to 0 \end{cases} \Longrightarrow \quad v \in (I^{0,0}_\sigma \cap) \ \underline{\ker N}$$

$$\Longrightarrow \quad \begin{array}{ccc} N u_n & = & N w_n \\ \cap & & \cap \\ F_n^{-1} & \cap & W_{-3} = \{0\} \end{array} \Longrightarrow \text{ both } = 0.$$

Step 4 : $\{u_n\} \subset$ finite set ⊛

Fix $n_0 \in \mathbb{N}$ and set

$$\underline{\mathcal{F}_n^\bullet} := e^{(z_{n_0} - z_n)N} F_n^\bullet = e^{(z_{n_0} - \cancel{z_n})N} \cancel{e^{z_n N}} e^{\Gamma(s_n)} F^\bullet_{\lim} \xrightarrow[n\to\infty]{} e^{z_{n_0} N} F^\bullet_{\lim} =: \underline{\mathcal{F}^\bullet_\infty} \in D.$$

($e^{\Gamma(s_n)}$: $e^{2\pi i z_n}$)

By step 3, $u_n \in F^0_n \Rightarrow u_n \in \mathcal{F}^0_n \underset{u_n \text{ real}}{\Longrightarrow} u_n$ type $(0,0)$ for $\varphi_{\mathcal{F}_n}$

$$\Longrightarrow \quad K^{1/2} \geq Q(u_n, u_n)^{1/2} = Q(\varphi_{\mathcal{F}_n}(i) u_n, \overline{u_n})^{1/2} = \|u_n\|_{\mathcal{F}_n} \geq c_0' \|u_n\|_{\mathcal{F}_\infty} \quad (\forall n)$$

(reasoning as in Step 1)

$\Longrightarrow \quad \{u_n\} \subset$ finite set.

$u_n \in V_{\mathbb{Z}}$

⊛ For any $F^\bullet \in D$, $Q(\cdot,\cdot)$ is positive on the (discrete) set $Hg_F^{\mathbb{Z}} = F^0 \cap V_{\mathbb{Z}}$, and so $Hg_F^{\mathbb{Z}}(K) = \{u \in Hg_F^{\mathbb{Z}} \mid Q(u,u) \leq K\}$ is finite. But Prop. 4 isn't that simple, because u_n isn't in a fixed F^0, but rather in F^0_n, and $Hg^{\mathbb{Z}}_{F_n}(K)$ could grow with n for all we know! That is why this is "Step 4" and not "Step 1"!

Step 5:

Since $e^{-zN}\tilde{\Phi}(z) \left(= e^{\Gamma(z)}F^{\bullet}_{\lim}\right) \longrightarrow F^{\bullet}_{\lim}$ as $\mathrm{Im}(z)\to\infty$,

$$e^{-z_n N}F^0_n \longrightarrow F^0_{\lim}.$$

Since $\underset{\text{Step 3}}{u_n = e^{-z_nN}}u_n \in e^{-z_nN}F^0_n \cap W_0 \cap V_{\mathbb{Z}}$,

$$\underset{\mathbb{P}V_{\mathbb{R}}}{[u_n]} \to \ell' \subset F^0_{\lim} \cap W_0 \cap V_{\mathbb{R}}.$$

By Step 4, $u_n \in \ell'$ for $n \gg 0$, hence

$$u_n \in F^0_{\lim} \cap W_0 \cap V_{\mathbb{R}} \subset I^{0,0}_{\lim}$$

[and (for $n, n' \gg 0$) $u_n - u_{n'} = w_n - w_{n'} \in W_{-1}$ must be zero]. $\square$

We conclude with an application to 1-parameter VHS.

Let $\bar{\mathcal{S}}$ be a complete curve, $\mathcal{V} \to \mathcal{S} := \bar{\mathcal{S}} \setminus \{p_1, \ldots, p_m\}$ a $\mathbb{Z}$PVHS with monodromy group Γ, and $\Phi : \mathcal{S} \to \Gamma\backslash D$ the associated period map. Note that $\Gamma\backslash D$ parametrizes Γ-equivalence classes of Hodge structures (or flags) on a <u>fixed</u> vector space V (= reference fiber at some $p_0 \in \mathcal{S}$). Write $T_i \left(= T_i^{ss}T_i^{un}\right)$ for the local monodromies and $W^{p_i}_{\bullet} = W(N_i)$, $\left(N_i = \log T_i^{un}\right)$ for the local monodromy weight filtrations.

<u>Proposition 5</u>: Suppose T_1 is of infinite order and primitive in Γ ($\nexists\, \gamma \in \Gamma$ with $\gamma^{\mu} = T_1$ and $\mu > 1$), and that the LMHS

at p_1 is distinct(*) from that at $p_2, \ldots, p_m$. Then $\overline{\Phi}$ is injective off a finite set.

Proof: Consider the exterior-tensor-product VHS $\mathcal{V} \boxtimes \mathcal{V}^* \to \mathcal{S} \times \mathcal{S}$ with local system fiber $\mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z},s'}, V_{\mathbb{Z},s})$ over (s,s'), with period map to $\Gamma\backslash D \times \Gamma\backslash D$. (We are using Q to identify Hodge structures on V & V^*.) The diagonal is precisely the locus where the Hodge flags are Γ-related, viz. $\gamma F^\bullet = {}'F^\bullet$ on $V_{\mathbb{C}}$, and its preimage is the Hodge locus $\mathcal{S}(\mathrm{id}_V)$ of $\mathrm{id}_V \in \mathrm{End}_{\mathbb{Z}}(V_{\mathbb{Z}})$. Obviously $\mathcal{S}(\mathrm{id}_V) \supset \Delta_{\mathcal{S}}$.

Now suppose $\overline{\Phi}$ is NOT injective off a finite set: then $\exists$ sequences $s_1, s_2, \ldots$ & $s_1', s_2', \ldots$ (all distinct) s.t. $\overline{\Phi}(s_i) = \overline{\Phi}(s_i')$ $(\forall i)$, i.e. distinct $(s_i, s_i') \in \mathcal{S}(\mathrm{id}_V) \setminus \Delta_{\mathcal{S}}$. Since $\mathcal{S}(\mathrm{id}_V)$ is algebraic, either (a) $\mathcal{S}(\mathrm{id}_V) = \mathcal{S} \times \mathcal{S}$ or (b) $\mathcal{S}(\mathrm{id}_V)$ contains a 1-dimensional component distinct from $\Delta_{\mathcal{S}}$.

If (a) holds, then $\mathcal{V}$ is isotrivial, and the LMHS are all pure and Γ-related, in contradiction to our hypothesis.

If (b) holds, then $\overline{\mathcal{S}(\mathrm{id}_V)}$ contains a point of the form (p_1, p'). By Prop. 3(b) (and taking global monodromy into account),

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(*) technically, the LMHS is only defined up to the action of $e^{\mathbb{C}N}$ on $F^\bullet_{\lim}$, because rescaling the local coordinates has this effect. Moreover, globally speaking, we cannot distinguish LMHS which are isomorphic via the action of Γ. Though this is stronger than needed, if T_1^{un} (resp. N_1) has a different Jordan block structure than $T_2^{un}, \ldots, T_m^{un}$ (resp. $N_2, \ldots, N_m$), that would certainly provide the required distinction.

$\gamma(\circ\, \mathrm{id}_V) \in F^0_{\lim,(p_1,p')} \cap W_0^{(p_1,p')} \cap \mathrm{End}(V_{\mathbb{Z}}) = \mathrm{Hom}_{\underline{\underline{MHS}}}((V, F^\bullet_{\lim,p'}, W^{p'}_\bullet), (V, F^\bullet_{\lim,p_1}, W^{p_1}_\bullet))$.

That is, γ gives an isomorphism between the 2 LMHS. If $p' \in \{p_2, \ldots, p_m\}$, this contradicts the hypothesis; if $p' \in C$ then $W^{p'}_\bullet$ is trivial whereas the infinite-order assumption on Γ_1 means $W^{p_1}_\bullet (= W(N_1)_\bullet)$ is not, again a contradiction. Finally, if $p' = p$, and $U \subset \bar{C}$ denotes a neighborhood of p_1, $\bar{\Phi}$ has degree $d > 1$ on $U \setminus \{p_1\}$, hence (being holomorphic) factors

$$\begin{array}{ccccc} \Delta^* \cong U\setminus\{p_1\} & \longrightarrow & \Delta^* & \xrightarrow{\iota} & \Gamma\backslash D \\ z & \longmapsto & z^d & & \end{array}.$$

Taking T to be ι_* of a generator of $\pi_1(\Delta^*)$, we have $T_1 = T^d$ in contradiction to primitivity. $\square$

<u>Remark</u>: The theory of "partial compactifications" of $\Gamma\backslash D$ due to Kato & Usui provides an alternate proof of the above (basically via the Hausdorffness of their construction). //