7. Absolute Hodge classes and Voisin's strategy

Let $X/k \subset \mathbb{C}$ be a smooth projective variety. The action of $\sigma \in \mathrm{Aut}(\mathbb{C})\,(=\mathrm{Gal}(\mathbb{C}/\mathbb{Q}))$ on coefficients of its defining equations yields a new variety ${}^{\sigma}X$ (defined over $\sigma(k)$). Identifying de Rham cohomology with algebraic differential forms

$$(1) \qquad F^p H^{2p}_{(dR)}(X^{an}_{\mathbb{C}}, \mathbb{C}) \cong \mathbb{H}^{2p}_{Zar}(X_{\bar{k}}, \Omega^{\bullet \geq p}_{X_{\bar{k}}}) \otimes_{\bar{k}} \mathbb{C}$$

and letting σ act on everything on the RHS (including the defining equations of X, and including the "$\mathbb{C}$") induces

$$(2) \qquad \sigma_* : F^p H^{2p}(X^{an}_{\mathbb{C}}, \mathbb{C}) \to F^p H^{2p}(({}^{\sigma}X)^{an}_{\mathbb{C}}, \mathbb{C}).$$

Recall that the ($\mathbb{Q}$-) Hodge classes are given by $Hg^p(X)_{\mathbb{Q}} = H^{2p}(X,\mathbb{Q}) \cap F^p H^{2p}(X,\mathbb{C})$.

Definition 1: The absolute Hodge classes are defined by

$$AHg^p(X) := \{\xi \in Hg^p(X)_{\mathbb{Q}} \mid \sigma_*(\xi) \in Hg^p({}^{\sigma}X)_{\mathbb{Q}} \ (\forall \sigma \in \mathrm{Aut}(\mathbb{C}))\}.$$

We say ξ is weakly absolute if $\sigma_*(\xi) \in Hg^p({}^{\sigma}X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ $(\forall \sigma)$. //

Since the conjugates remain F^p in (2), the issue is whether they remain $\mathbb{Q}$-rational.

Theorem 1: Classes of algebraic cycles are absolute:

$$\mathrm{im}(cl^p_X) \subset AHg^p(X).$$

In what follows we assume k algebraically closed for simplicity.

Idea of proof: We may Zariski-locally think of irreducible components of a cycle as complete intersections, viz.

$$Z_U = Z \cap U = \bigcap_{i=1}^{p} \{ f_i^U = 0 \}.$$

Then the elements (writing $d\log f := \frac{df}{f}$ and $\Omega^{p,c} := \ker(d) \subset \Omega^p$)

$$(3) \qquad \bigwedge_{i=1}^{p} d\log f_i^U \in \operatorname{image}\Big\{ H^0(U \setminus \bigcup_i \{f_i^U = 0\}, \Omega^{p,c}_{U/k}) \to \underbrace{H^{p-1}_{Zar}(U \setminus Z_U, \Omega^{p,c}_{U/k})}_{\substack{\text{think: Čech cohom.} \\ \text{w.r.t. cover } \{U \setminus f_i = 0\}_i \\ \text{of } U \setminus Z_U}} \xrightarrow{\delta} H^p_{Z_U, Zar}(U, \Omega^{p,c}_{U/k}) \Big\}$$

— which are independent of choices —

yield a global section of the cohomology sheaf

$$\mathcal{H}^p_Z(\Omega^{p,c}_{X/k}) = \mathbb{H}^{2p}_Z(\Omega^{p,c}_{X/k}[-p]) \longrightarrow \mathbb{H}^{2p}_Z(\Omega^{\bullet \geq p}_{X/k}),$$

hence an element of

$$H^0(X, \mathbb{H}^{2p}_Z(\Omega^{\bullet\geq p}_{X/k})) \xleftarrow[\cong]{\text{edge hom.}} \mathbb{H}^{2p}_{Z, Zar}(X, \Omega^{\bullet \geq p}_{X/k}) \xrightarrow{\text{Gy}} \mathbb{H}^{2p}_{Zar}(X, \Omega^{\bullet \geq p}_{X/k}).$$

Then one has to check that this identifies with the cycle-class map.

To me, this approach is too complicated (too much to check, too messy, etc.).

Instead, it's better to accept a bit more abstraction in order to get clarity.

Actual proof: Introduce the Milnor K-group of a field $\mathbb{F}$:

$$K_n^M(\mathbb{F}) := \frac{\mathbb{F}^* \wedge \cdots \wedge \mathbb{F}^*}{\langle f, 1-f \rangle} := \frac{\text{free abelian group generated by symbols } \{f_1, \ldots, f_n\}}{\text{relations} \begin{cases} \{\ldots, f_i g_i, \ldots\} = \{\ldots, f_i, \ldots\} + \{\ldots, g_i, \ldots\} \\ \{\ldots, f, g, \ldots\} = -\{\ldots, g, f, \ldots\} \\ \{\ldots, f, \ldots, 1-f, \ldots\} = 0 \end{cases}}$$

and sheaves $\mathcal{K}^M_{n,X} := \frac{\mathcal{O}_X^* \wedge \cdots \wedge \mathcal{O}_X^*}{\langle f, 1-f \rangle}$. We have for each subvariety Y of codim. 1 in X the Tame symbol (or K-theoretic residue) map

$$(4) \qquad \operatorname{Tame}_Y : K_n^M(k(X)) \longrightarrow K_{n-1}^M(k(Y))$$

given by

(a) writing any element in LHS (4) as a sum of symbols $\{f_1, \ldots, f_n\}$ in which only f_1 is allowed to have nonzero degree along Y; (*)

(b) setting Tame_Y of such a symbol to be $\deg_Y(f_1)\{f_2, \ldots, f_n\}|_Y$.

Exercise 1: Writing $\delta_Y = \imath^Y_*(1)$ for the current of integration over Y, and $d[\cdot]$ for exterior derivative on currents (defined by integration by parts, cf. Griffiths–Harris §3), check that

(i) $d\left[\frac{dz}{z}\right] = 2\pi i\,(\delta_{\{0\}} - \delta_{\{\infty\}})$ on $\mathbb{P}^1$

(ii) $d\left[\frac{df}{f}\right] = 2\pi i \sum_{Y \subset X \text{ cd. } 1} \deg_Y(f)\,\delta_Y$.

(iii) $d\left[\frac{df}{f} \wedge \omega\right] = 2\pi i \sum_{Y \subset X \text{ cd. } 1} \deg_Y(f)\,\imath^Y_* \omega$.
$C_{\text{smooth/closed}}$

Conclude that (iv) $d[\hat{\Omega}] = 2\pi i \sum_{Y \subset X \text{ cd. } 1} \imath^Y_* \mathrm{Res}_Y \hat{\Omega}$ if $\hat{\Omega}$ is a form with log poles.

[Note: $K_0^M(\mathbb{F}) = \mathbb{Z}$, $K_1^M(\mathbb{F}) = \mathbb{F}^*$.]

Next, we have the Gersten resolution

$$(5) \qquad \mathcal{K}^M_{p,X} \longrightarrow \mathcal{G}^\bullet_{p,X}\,,$$

where $\mathcal{G}^\bullet_{p,X} := \coprod_{x \in X^k} K^M_{p-k}(k(x))$ as skyscraper sheaf on $\bar{x}^{\mathrm{Zar}}$, $X^k \subset$ {codim. k points}, with differential D given by Tame symbols along all codim $k+1$ points.

which is acyclic (in the Zariski topology). So

$$(6) \qquad H^p_{\mathrm{Zar}}(X, \mathcal{K}^M_{p,X}) \cong \mathbb{H}^p(X, \mathcal{G}^\bullet_{p,X}) \cong \frac{\Gamma(X, \mathcal{G}^p_{p,X})}{D(\Gamma(X, \mathcal{G}^{p-1}_{p,X}))} = \frac{\bigoplus_{x \in X^p} K_0^M(k(x)) (\cong \mathbb{Z}_{\bar{x}})}{D\left(\bigoplus_{y \in X^{p-1}} K_1^M(k(y)) (\cong k(y)^*)\right)}$$

$$= \frac{Z^p(X_k)}{\sum_{y \in X^{p-1}} \mathrm{Div}(k(y)^*)} = CH^p(X_k)\,,$$

and we obtain a map of complexes

(*) that this is always possible is a consequence of the Bloch moving lemma, which we will cover in Part II.

(7)
$$\begin{tikzcd}[column sep=small]
\mathcal{K}^M_{p,X} \arrow[r] \arrow[d, "\Lambda^p d\log\ (\text{on regular fcns.})"] & K^M_p(k(X)) \arrow[r] \arrow[dd, "\Lambda^p d\log\ (\text{on rational fncts})"] & \coprod_{x\in X^1} K^M_{p-1}(k(x)) \arrow[r] \arrow[dd, "2\pi i \sum \iota^{\bar{x}}_* \Lambda^{p-1} d\log"] & \coprod_{x\in X^2} K^M_{p-2}(k(x)) \arrow[r] \arrow[dd, "(2\pi i)^2 \sum \iota^{\bar{x}}_* \Lambda^{p-2} d\log"] & \cdots \arrow[r] & \coprod_{x\in X^{p-1}} k(x)^* \arrow[r] \arrow[dd, "(2\pi i)^{p-1} \sum \iota^{\bar{x}}_* d\log"] & \coprod_{x\in X^p} \mathbb{Z}_{\bar{x}} \arrow[dd, "(2\pi i)^p \sum \iota^{\bar{x}}_*"] \\
\Omega^{\bullet\geq p}_{X/k}[p] \arrow[d, "\int"] & & & & & & \\
\Omega^{\bullet\geq p}_{X/\mathbb{C}}[p] \arrow[r] & F^pD^p_X \arrow[r, "d"] & F^pD^{p+1}_X \arrow[r, "d"] & F^pD^{p+2}_X \arrow[r] & \cdots \arrow[r] & F^pD^{2p-1}_X \arrow[r] & F^pD^{2p}_X
\end{tikzcd}$$

$\leftarrow$ Zariski closure

in which: • "$\Lambda^k d\log$" sends $\{f_1, \ldots, f_k\} \mapsto \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_k}{f_k}$;

- the bottom row expands the resolution $\Omega^{\bullet\geq p}_X[p] \to F^pD^\bullet[p]$;
- the right-most vertical map sends a section $Z \in Z^p(X_k)$ to the current of integration over Z (which was how we defined cl^p_X) times $(2\pi i)^p$; and
- Exercise 2: Use Exercise 1 to check the diagram commutes! (Also, check that $\Lambda^k d\log$ is well-defined on Milnor K-groups!)

This $\leftarrow$ i.e. (7) induces a

(8)
$$\begin{tikzcd}
H^p_{Zar}(X_k, \mathcal{K}^M_{p,X_k}) \arrow[d, "\Lambda^p d\log"] & CH^p(X_k) \arrow[l, "\cong"'] \arrow[dl, "=: cl^{p,alg}_X"] \arrow[ddl, bend left, "(2\pi i)^p cl^p_X"] & Z^p(X_k) \arrow[l, two heads] \arrow[dd, "Z \mapsto (2\pi i)^p \delta_Z"] \\
\mathbb{H}^{2p}_{Zar}(X_k, \Omega^{\bullet\geq p}_{X/k}) \arrow[d, "\otimes_k \mathbb{C}"] & & \\
\mathbb{H}^{2p}(X_\mathbb{C}, \Omega^{\bullet\geq p}_{X/\mathbb{C}}) & F^pH^{2p}(X^{an}_\mathbb{C}, \mathbb{C}) \arrow[l, "\cong"'] & F^pD^{2p}(X) \arrow[l, two heads]
\end{tikzcd}$$

which verifies that the algebraic cycle-class map (defined by the dotted composition) matches $(2\pi i)^p cl^p_X$. In fact, we often replace cl^p_X by $(2\pi i)^p cl^p_X$ for this reason, writing $Hg^p(X)$ for

(9)
$$H^{2p}(X, \mathbb{Q}(p)) \cap F^p H^{2p}(X, \mathbb{C})$$

where $(\mathbb{Q}(p) = (2\pi i)^p \mathbb{Q})$. (We will do this below.)

Now to finish the proof, given $Z \in Z^p(X_k)$ and $\sigma \in Aut(\mathbb{C})$, and noting that $\{f_1, \dots, f_p\} \mapsto \frac{df_1}{f_1} \wedge \dots \wedge \frac{df_p}{f_p}$ commutes with σ $\Rightarrow$ $cl_X^{p,alg}$ does, we have

(commutativity of (8))

$$(10) \qquad \sigma_* \, cl_X^p(Z) \underset{\uparrow}{=} {}^\sigma cl_X^{p,alg}(Z) = cl_{\sigma X}^{p,alg}({}^\sigma Z) = cl_{\sigma X}^p({}^\sigma Z).$$

(by commutativity of (8) & definition of σ_* in (2))

So if $\xi \in Hg^p(X)$ is $cl_X^p(Z)$, then $\sigma_* \xi \in cl_{\sigma X}^p(CH^p({}^\sigma X)) \subset Hg^p({}^\sigma X)$, which is what we wanted to show. $\square$

The main points in the proof are:

- writing Chow groups as K-cohomology and taking the "dlog map" abstractifies the previous "idea of proof"
- the dlog map is purely algebraic so intertwines Galois conjugation
- the diagram (8) shows that the dlog map matches cl_X^p under the isomorphism (1) when applied to a cycle
- therefore cl_X^p intertwines the map σ_* defined via (1). //

Theorem 1 gives rise to the following picture:

$$(11) \qquad cl_X^p(Z^p(X)) \subset \overbrace{AHg^p(X) \subset Hg^p(X)}^{HC}$$

AHC: "Hodge $\Rightarrow$ AH"

The (weaker) AHC has one of the same consequences as the HC. Let $\pi : \mathcal{X} \to \mathcal{S}$ be a smooth proper morphism of varieties defined$/k$ (k f.g.$/\mathbb{Q}$), giving rise via $\mathbb{V}_{\mathbb{Z}} := R^n \pi_* \mathbb{Z}$ to a period map

$\bar{\Phi} : \mathcal{S} \to {}_{\Gamma}\backslash D$. Let $D_M \subset D$ be a M-T subdomain, Γ_M its stabilizer in Γ.

Theorem 2: If AHC holds, then any irreducible component $\mathcal{Y}$ of $\bar{\Phi}^{-1}(\Gamma_M \backslash D_M)$ is defined over $\bar{k}$. [compare Prop. I.B.6.1(b) (and its Corollary)]

Sketch: By the CDK result, $\mathcal{Y}$ is algebraic. Consider the (irreducible) $\bar{k}$-spread $\mathcal{L} \subset \mathcal{S}$ of any $p \in \mathcal{Y}(\mathbb{C})$, i.e. the Zariski closure of the set $\{q \in \mathcal{S}(\mathbb{C}) \mid X_q = {}^{\sigma}X_p \text{ for some } \sigma \in \mathrm{Aut}(\mathbb{C}/\bar{k})\}$. The $\{\sigma\}$ produce a continuous family of isomorphisms $H^n_{dR}(X_p) \overset{\to}{\cong} H^n_{dR}(X_q)$, inducing (by AHC) isomorphisms defined $/\mathbb{Q}$ of spaces of Hodge tensors. It follows that these Hodge-tensor-spaces are constant with respect to the $\mathbb{Q}$-Betti structure; that is, $\mathcal{L} \subset \mathcal{Y}$, and so the $\bar{k}$-spread of $\mathcal{Y}$ is $\mathcal{Y}$. $\square$

Corollary: If $n = 2p$ and $\nu \in \mathbb{V}_{s_0}$ is absolute Hodge at some point, e.g. s_0, then the Hodge locus $\mathcal{S}(\nu)$ is defined $/\bar{k}$.

As a consequence, Voisin observes that we can break the HC into two steps:

(i) proving that Hodge classes are (weakly) absolute; and

(ii) proving that (weakly) absolute classes on varieties def'd $/\bar{\mathbb{Q}}$ are classes of algebraic cycles.

For supposing we have $\xi \in Hg^p(X)$, spreading X out gives $\mathcal{X} \to \mathcal{S}/\bar{\mathbb{Q}}$. If ξ is (weakly) absolute, it parallel translates to a (weakly) absolute Hodge class on every fiber, a fortiori some fiber $/\bar{\mathbb{Q}}$.

I don't know of any case where (ii) is any easier than the HC, i.e. "cuts down the cycles to check HC for" in some meaningful way (e.g. see Deligne's result below). However, the following result illustrates that <u>when a Hodge locus is positive-dimensional</u> we do have some chance of proving the class is absolute (towards (i)):

VHS arises from $\pi : \mathcal{X} \to \mathcal{S}$ def'd $/k$

<u>Theorem 3 (Voisin, 2006)</u>: (Notation as above, with $n = 2p$.)

Suppose $\mathcal{T} \subset \mathcal{S}$ is an irreducible subvariety, defined $/\mathbb{C}$, such that:

(i) $\mathcal{T}$ is a component of the Hodge locus of some $\alpha \in (\mathcal{F}^p \cap \mathbb{V}_{\mathbb{Q}})_{t_0}$; and

(ii) $\pi_1(\mathcal{T}, t_0)$ fixes (under ∇-flat continuation in $\mathbb{V}_{\mathbb{C}}$) only the line generated by α.

Then $\mathcal{T}$ is defined over $\bar{k}$. Also: If $k = \mathbb{Q}$, then α is <u>weakly absolute</u> (see p. 1)

<u>Sketch</u>: Except in the trivial case, the hypotheses force $\dim(\mathcal{T}) > 0$. According to (ii), we may extend α to a ∇-flat family over $\mathcal{T}$. Given $\sigma \in \mathrm{Aut}(\mathbb{C}/\bar{k})$, ${}^{\sigma}\alpha$ is a ∇-flat family over ${}^{\sigma}\mathcal{T}$, by

algebraicity* of ∇. Moreover, the fixed part of $\mathbb{W}_{\mathbb{C}}$ over ${}^{\sigma}\mathcal{T}$ must be of rank 1, since otherwise (applying σ^{-1} and algebraicity of ∇) its fixed part over $\mathcal{T}$ could not satisfy (ii). So ${}^{\sigma}\alpha = \lambda\beta$, where $\lambda \in \mathbb{C}$ and β is $\mathbb{Q}$-Betti; but then β is Hodge, since ${}^{\sigma}\alpha \in F^p$.

As in the proof of Theorem 2, varying σ yield a continuum of conjugates ${}^{\sigma}\mathcal{T}$ on which the line $\mathbb{C}\langle {}^{\sigma}\alpha \rangle$ remains rational; hence it is constant. Since the polarization is algebraic, $Q(\alpha, \alpha) = Q({}^{\sigma}\alpha, {}^{\sigma}\alpha) = \lambda^2 Q(\beta, \beta) \implies \lambda^2 \in \mathbb{Q}$, and again by continuity $\lambda = 1$. Therefore ${}^{\sigma}\alpha$ remains Hodge, and α extends to a Hodge class on the $\bar{k}$-spread of $\mathcal{T}$, which must then (by (i)) be $\mathcal{T}$ itself. Also: If $k = \mathbb{Q}$, and $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ (wolog $\mathrm{Aut}(\bar{\mathbb{Q}}/\mathbb{Q})$), we at least get ${}^{\sigma}\alpha = \lambda\beta$ with $\lambda^2 \in \mathbb{Q} \implies \lambda \in \bar{\mathbb{Q}}$; so α is weakly absolute. $\square$

An amazing result, which is at least the equal of C-D-K as evidence for the HC, is the following one, which Deligne needed for his construction of canonical models (over a number field) for Shimura varieties of Hodge type:

Theorem 4 (Deligne, 1982): AHC holds for abelian varieties.

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* on alg. dR cohomology, it is induced by the connecting homomorphism assoc. to $0 \to \pi^* \Omega^1_{\mathcal{S}} \wedge \Omega^{\bullet \geq p-1}_{\mathcal{X}}[-1] \to \Omega^{\bullet \geq p}_{\mathcal{X}} \to \Omega^{\bullet \geq p}_{\mathcal{X}/\mathcal{S}} \to 0$.

Idea of proof: (1) Start with a family $\mathcal{A} \to \mathcal{S}$ of abelian varieties over a connected Shimura variety of Hodge type.

(2) CM points are dense in $\mathcal{S}$. Principle B (algebraicity of ∇) allows us to reduce AHC for a generic A_s to AHC for CM abelian varieties.

(3) Principle A (fixed tensors of the "absolute MTG" are absolute Hodge) reduces us to finding absolute Hodge tensors cutting out the usual MTG, forcing "absolute MTG" = "usual MTG".

(4) Weil Hodge classes are absolute. (Together with "obvious" absolute Hodge classes, this does the job.) $\square$

To conclude §I.B, we want to mention a few more recent cases when the HC has been verified.

(A) The work of Bergeron – Millson – Moeglin

Let X be a connected compact Shimura variety of the form

(identity conn. component)

$\Gamma\backslash G_{\mathbb{R}}^{\oplus}/K$, where $G_{\mathbb{R}} = O(p,2)$ or $U(p,1)$ (and $\dim_{\mathbb{C}} X = p$).

X has special cycles arising from divisors and sub-Shimura varieties arising from subgroups $O(p-k,2)$ resp. $U(p-k,1)$ (and intersection products of these).

<u>Theorem (BMM 2011, 2013)</u>: $Hg^n(\bar{X})$ is spanned by $cl^n_{\bar{X}}$ of the special cycles for $n \leq \frac{p}{3}$ (or $\geq \frac{2p}{3}$). [They have also shown this in the noncompact case for $G_{\mathbb{R}} = O(p,2)$]

Their results are proved by sophisticated techniques in the theory of automorphic forms, but the core of the argument is to show surjectivity of the <u>theta lift</u>

$$(12) \qquad \theta_* : H^{2n}(\mathfrak{g}, K; W) \to H^{2n}(\bar{X}, \mathbb{C})$$

onto the (n,n) part of cohomology and argue that certain classes on the left map to the classes of special cycles. Here W is the (∞-dim'l) <u>Weil representation</u> with underlying space the Schwartz functions $\mathcal{S}(V_{\mathbb{R}})$ on the orthogonal vector space (V, Q) on which $G^+_{\mathbb{R}} = O(p,2)^+$ acts. Given a cocycle $\varphi \in (\mathcal{S}(V_{\mathbb{R}}) \otimes \Lambda^{2n}\mathfrak{p}^*)^K$ (with class in LHS(12); here $\mathfrak{p} \oplus \mathfrak{k} = \mathfrak{g}$ is the Cartan decomposition), we consider $\bar{X}$ to parametrize a choice of lattice $\mathcal{L}_x \subset V_{\mathbb{R}}$ and send φ to the differential $2n$-form on $\bar{X}$ defined by $\sum_{\ell \in \mathcal{L}_x} \varphi(\ell)$. (The real power of this construction comes when one notices that there is an action of $SL_2(\mathbb{R})$ on $\mathcal{S}(V_{\mathbb{R}})$ that commutes with $G^+_{\mathbb{R}}$'s, and which intertwines (12).)

(B) <u>A result of Arapura</u>

I'll have nothing to say about this, since I thought it might be a good paper for one of you to present as your final project.

<u>Theorem (Arapura 2013)</u>: Let $Y \to \mathbb{P}^n$ be a $p:1$ cyclic cover (p prime) branched over a hyperplane arrangement D with normal crossings. Take $X \twoheadrightarrow Y$ to be a toroidal desingularization. Then the HC holds for X.

(C) <u>Yun's construction</u>

Let $G = G_2, E_7, E_8$ and $\mathcal{S} = \mathbb{P}^1 \setminus \{0,1,\infty\}$. Using techniques from the Geometric Langlands program, Yun (Inventiones 2014) constructs functors (roughly speaking)

(what actually appears in Yun's paper)

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{representations}\\ \rho : G \to \mathrm{Aut}(V)\end{array}\right\} \arrow[r, "\Psi"] \arrow[d, "\Theta"'] \arrow[rr, dashed, bend left] & \left\{\begin{array}{c}\text{VHS } \mathcal{V}/\mathcal{S}\\ \text{with MTG } G\end{array}\right\} \arrow[r] & \left\{\begin{array}{c}G\text{-local systems}\\ \mathbb{W} \to \mathcal{S}\end{array}\right\} \\
\left\{\begin{array}{c}\text{families of motives}\\ \mathcal{M} \underset{\pi}{\to} \mathcal{S}\end{array}\right\} \arrow[ur, "H^*"'] \arrow[urr, dotted] & &
\end{tikzcd}$$

Given any representation (ρ, W) of G which contains a copy of the trivial representation [such as $V^{\otimes 3}$ (V = 7-dim'l irrep.) for $G = G_2$], we get a morphism $(\text{triv.}, \mathbb{Q}) \to (\rho, W)$. Applying Ψ gives a morphism $\mathbb{Q}(-*) \hookrightarrow H^{2*}(\mathcal{M}/\mathcal{S})$ of VHS*, hence a family of Hodge classes; applying Θ gives a morphism

$$\begin{tikzcd}
\mathcal{M}_{\text{triv.}} \arrow[rr] \arrow[dr] & & \mathcal{M}_\rho \arrow[dl] \\
& \mathcal{S} &
\end{tikzcd}$$

whose image produces the needed cycle.** In this way all Hodge tensors of

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*/** this statement is not rigorously formulated and has not been checked carefully, so should not be quoted.

any V_ρ over all of $\mathcal{S}$ should automatically correspond to algebraic cycles. The catch is that, say, the cycles don't lie on the variety one would have hoped [e.g. one doesn't have a correspondence between $M_{\rho^{\otimes 3}}$ and "$M_\rho^{\times 3}$" in the G_2 example just mentioned]. So in some sense one gets the HC; in some sense one doesn't.

<u>Addendum</u>. The G_2 example in (C) is related to another "weird" case of the HC (wide open for the most part). Let $\varphi \in D = M(\mathbb{R})/H_\varphi$ be a HS and consider the corr. HS $\mathrm{Ad}\circ\varphi$ on the Lie algebra $\mathfrak{m}$. Now

- M $\mathbb{Q}$-algebraic $\Rightarrow$ $[\ ,\ ] : \mathfrak{m}\otimes\mathfrak{m} \to \mathfrak{m}$ is rational
- $[\mathfrak{m}^{(a,-a)}, \mathfrak{m}^{(b,-b)}] \subset \mathfrak{m}^{(a+b,-(a+b))}$ $\Rightarrow$ $[\ ,\ ]$ is "Hodge"

$$\Rightarrow \quad [\ ,\ ] \in Hg^{1,2}\mathfrak{m} \ \left(\subset \mathfrak{m}\otimes(\mathfrak{m}^\vee)^{\otimes 2}\right)$$

If $(\mathfrak{m}, \mathrm{Ad}\circ\varphi, -B)$ comes from $H^n(X)$ one has the

$\boxed{\text{Lie-bracket Hodge Conjecture: } [\ ,\ ] \text{ comes from an algebraic cycle on } X\times X\times X.}$

Exercise: $(M = SL_2)$ As a PHS, one can identify $\mathfrak{m} \cong End(H^1(E), \mathbb{Q})$ (non-CM ell. curve) $\cong H^2_{tr}(E\times E)$. Taking $X = E\times E$, find the cycle in $CH^3(X^{\times 3})$ giving $[\ ,\ ]$.