C. The Abel-Jacobi map and normal functions

1. Extensions of MHS and the Deligne cycle-class

The "Betti" cycle-class map cl_X^p sufficed for our discussion of the Hodge conjecture, but it merely skims over the jungle of algebraic cycles: e.g. for 0-cycles, it knows nothing about the cycles of degree zero! To go further, we need some generalities on Ext groups (done more thoroughly in my Hodge Theory notes, §IV.C).

Let $\mathcal{C}$ be an abelian category with "enough injective objects", and (for $X \in \mathcal{C}$) define $\mathrm{Ext}^i_{\mathcal{C}}(X,-) := R^i\mathrm{Hom}_{\mathcal{C}}(X,-)$. Given $A \hookrightarrow A^\bullet$ an injective resolution (in $\mathcal{C}$), we have

$$(1) \qquad \mathrm{Ext}^1_{\mathcal{C}}(X,A) = \frac{\ker\{\mathrm{Hom}_{\mathcal{C}}(X,A^1) \to \mathrm{Hom}_{\mathcal{C}}(X,A^2)\}}{\mathrm{Im}\{\mathrm{Hom}_{\mathcal{C}}(X,A^0) \to \mathrm{Hom}_{\mathcal{C}}(X,A^1)\}}$$

$$\cong \frac{\mathrm{Hom}_{\mathcal{C}}(X,K)}{\text{Homs factoring thru } A^0} \qquad [\text{taking } K := \ker(A^1 \to A^2)]$$

$$\cong \frac{\text{Short-exact seq's } A \to E \overset{\pi}{\to} X}{\text{split s.e.s.'s}}$$

$$\text{split s.e.s.'s} \Updownarrow \left[\begin{array}{c}\exists \text{ section } s : X \to E \\ \text{s.t. } \pi \circ s = \mathrm{id}_X\end{array}\right]$$

[reason as follows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & A^0 \arrow[r] & K \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u, equal] & E \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "\phi"'] & 0 \end{tikzcd}$$

$E = \ker\{A^0 \oplus X \to K\}$

split $\iff$ ϕ factors thru some $\tilde{\phi} : X \to A^0$ (take $X \to E$ given by $(x, -\tilde{\phi}(x))$]

$$\cong \frac{\text{"extensions"}}{\text{"congruence"}} \leftsquigarrow [\text{morphisms of extensions which are identity on outer terms}]$$

with abelian group structure given by "Baer summation". We also have, for any s.e.s. $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, a long-exact sequence

(2)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X,A) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X,C) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^1_{\mathcal{C}}(X,A) \arrow[r] & \mathrm{Ext}^1_{\mathcal{C}}(X,B) \arrow[r] & \mathrm{Ext}^1_{\mathcal{C}}(X,C) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^2_{\mathcal{C}}(X,A) \arrow[r] & \cdots &
\end{tikzcd}$$

and the theory shows that

(3) if $\mathrm{Ext}^1_{\mathcal{C}}(X,-)$ is right-exact, then all higher $\mathrm{Ext}^i_{\mathcal{C}}(X,-)$ $(i \geq 2)$ vanish.

Specializing to $\mathcal{C} = \mathrm{MHS}^*$, if $A = (A_{\mathbb{Q}}, W_\cdot, F^\cdot)$ and $B = (B_{\mathbb{Q}}, W_\cdot, F^\cdot)$ are 2 MHS, $\mathrm{Hom}_{\mathrm{MHS}}(A,B)$ are the morphisms $\xi_{\mathbb{Q}} : A_{\mathbb{Q}} \to B_{\mathbb{Q}}$ with $\xi_{\mathbb{Q}} W_\cdot \subset W_\cdot$, $\xi_{\mathbb{C}} F^\cdot \subset F^\cdot$.

Exercise: Show that in fact ξ is then strictly compatible with $F^\cdot$ & $W_\cdot$. (Use the $I^{p,q}$'s).

We have (recalling $\mathbb{Q}(p)$ is the rank-1 HS of weight $-2p$ & type $(-p,-p)$)

(4)
$$\theta^0 : \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), A) \xrightarrow{\cong} W_0 A_{\mathbb{Q}} \cap F^0 W_0 A_{\mathbb{C}} =: Hg(A)$$
$$\boxed{\xi} \longmapsto \xi(1),$$

and

(5)
$$\theta^1 : \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), A) \xrightarrow{\cong} \frac{W_0 A_{\mathbb{C}}}{F^0 W_0 A_{\mathbb{C}} + W_0 A_{\mathbb{Q}}} =: J(A)$$

$\in$

$\left[\boxed{0 \to A \to E \to \mathbb{Q}(0) \to 0}\right]$

take $\begin{cases} \mu_{\mathbb{Q}} \in W_0 E_{\mathbb{Q}} \text{ mapping to } 1 \in \mathbb{Q}(0) \\ \mu_F \in F^0 W_0 E_{\mathbb{C}} \end{cases}$ $\longmapsto [\alpha]$

then $\exists$ unique $\alpha \in W_0 A_{\mathbb{C}}$ maps to $\mu_{\mathbb{Q}} - \mu_F$

Exercise: verify the $\cong$ in (5).

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* Category of mixed Hodge structures. Lacks "enough injectives", but (1)-(3) still hold by general theory due to Verdier & Yoneda. (cf. Peters-Steenbrink appendix)

Also, if $A \twoheadrightarrow B$ then $J(A) \twoheadrightarrow J(B)$ is clear; from this we see (by (3)) $\mathrm{Ext}^{\geq 2}_{\mathrm{MHS}} = 0$. More generally, $\mathrm{Hom}_{\mathrm{MHS}}(A,B) \cong \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), A^{\vee} \otimes B)$ and $\mathrm{Ext}^1_{\mathrm{MHS}}(A,B) \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), A^{\vee} \otimes B)$.

Remarks: (i) We could replace "MHS" by "PMHS" which means graded-polarized: i.e. $\exists\ Q_\ell : Gr^W_\ell A_{\mathbb{Q}} \otimes Gr^W_\ell A_{\mathbb{Q}} \to \mathbb{Q}$ $(\forall \ell)$ giving the $\{Gr^W_\ell A\}$ the structure of PHS.

Exercise:
$$\mathrm{Hom}_{\mathrm{PMHS}}(\mathbb{Q}(0), A) = \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), A)$$
$$\mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0), A) \cong \frac{W_{-1} A_{\mathbb{C}}}{W_{-1} A_{\mathbb{C}} \cap \{F^0 W_0 A_{\mathbb{C}} + W_0 A_{\mathbb{Q}}\}} \left(\hookrightarrow \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), A) \right).$$

(ii) One also has the notion of $\mathbb{Z}$-MHS, which includes $A_{\mathbb{Z}} \subset A_{\mathbb{Q}}$ (with $W_\bullet$ still def'd on $A_{\mathbb{Q}}$), and $W_0 A_{\mathbb{Q}} \cap A_{\mathbb{Z}}$ replaces $W_0 A_{\mathbb{Q}}$ in the definition of $J(A)$ (and in (5)), which is then a complex Lie group $\iff W_0 A = W_{-1} A$.
Exercise [Hint: draw the picture, and determine the conditions under which $W_0 A_{\mathbb{Z}}$ is a lattice in $W_0 A_{\mathbb{C}} / (W_0 \cap F^0) A_{\mathbb{C}}$]

To define the AJ (Abel-Jacobi) map, let X be a smooth projective algebraic variety of dim. n, $Z \in Z^p_{\mathrm{hom}}(X)$ a homologically trivial cycle. One way to say this is that there exists a chain
$$\Gamma \in C^{\mathrm{top}}_{2n-2p+1}(X) \quad \text{s.t.} \quad \partial\Gamma = Z_{\mathrm{top}}.$$
("top" stands for "topological", i.e. C^∞ chains in the analytic topology.) Alternatively, the class $[Z] := (2\pi i)^p [\delta_Z] \in H^{2p}_{|Z|}(X, \mathbb{Z}(p))$ dies under the map to $H^{2p}(X, \mathbb{Z}(p))$,

here I am computing cohom. w/ support on $|Z|$ by currents w/ support on $|Z|$

support of Z (= codim. p subscheme of X)

and so we can pull back the extension of MHS

$$\begin{tikzcd}
\left(H^{2p-1}_{|Z|}(X)\right) \arrow[d, equal] & & & & \\
0 \arrow[r] & H^{2p-1}(X)(p) \arrow[r] \arrow[d, equal] & H^{2p-1}(X\setminus|Z|)(p) \arrow[r] & \ker\left\{H^{2p}_{|Z|}(X)(p) \to H^{2p}(X)(p)\right\} \arrow[r] & 0 \\
\text{to get an extension} \;\; 0 \arrow[r] & H^{2p-1}(X)(p) \arrow[r] & \mathbb{E}_Z \arrow[r] \arrow[u, hook] & \mathbb{Q}\langle[Z]\rangle \arrow[r] \arrow[u, hook] \arrow[d, phantom, "\cong"] & 0 \\
 & & & \left(\mathbb{Q}(0)\right) &
\end{tikzcd}$$

in $\mathrm{Ext}^1_{\mathbb{Z}\text{-MHS}}\left(\mathbb{Z}(0), H^{2p-1}(X)(p)\right) \cong J\left(H^{2p-1}(X)(p)\right)$

(6) $$\underset{(\text{using }(5))}{\cong} \frac{H^{2p-1}(X,\mathbb{C})}{F^pH^{2p-1}(X,\mathbb{C}) + H^{2p-1}(X,\mathbb{Z}(p))} =: J^p(X) .$$

Explicitly, the element of $J^p(X)$ is given by taking the difference of lifts

$$\mu_{\mathbb{Z}} = [(2\pi i)^p \delta_\Gamma] \in H^{2p-1}(X\setminus|Z|, \mathbb{Z}(p)) \left(\cong H_{2n-2p+1}(X, |Z|; \mathbb{Z}(p-n))\right)$$

and

$$\mu_F = [\Xi] \in F^pH^{2p-1}(X\setminus|Z|, \mathbb{C}) \quad \text{with } \Xi \in F^pD^{2p-1}(X)$$

a current with $d[\Xi] = (2\pi i)^p \delta_Z \in F^pD^{2p}(X)$.

$(\Rightarrow (2\pi i)^p \delta_\Gamma - \Xi$ is d-closed$)$

$$\Rightarrow \theta'(\mathbb{E}_Z) = \left[(2\pi i)^p \delta_\Gamma - \Xi\right] \in J^p(X) .$$

Since $H^{2p-1}(X,\mathbb{C})/F^p \cong \left(F^{n-p+1}H^{2n-2p+1}(X,\mathbb{C})\right)^\vee$, we have

(7) $$J^p(X) \cong \left\{F^{n-p+1}H^{2n-2p+1}(X,\mathbb{C})\right\}^\vee \Big/ \mathrm{image}\left\{H_{2n-2p+1}(X,\mathbb{Z}(p))\right\} ;$$

i.e. $\theta'(\mathbb{E}_Z)$ defines a functional on closed C^∞ forms

$$\omega \in F^{n-p+1}A^{2n-2p+1}(X)$$

defined "up to periods". To compute it, write

$$\theta'(\mathbb{E}_Z)\,\omega = \int_X \left((2\pi i)^p\delta_\Gamma - \Xi\right)\wedge\omega \underset{\uparrow}{=} (2\pi i)^p \int_\Gamma \omega$$

$$\left\{\Xi \in F^p,\ \omega \in F^{n-p+1} \rightsquigarrow \Xi\wedge\omega \in F^{n+1}A^{2n}(X) = \{0\}\right\}$$

Definition 1: The AJ map is given by

$$AJ^p_X : CH^p_{hom}(X) \longrightarrow J^p(X)$$

$$Z = \partial\Gamma \longmapsto (2\pi i)^p \int_\Gamma (\cdot) \quad \{\text{in } (7)\}.$$

(The $(2\pi i)^p$ is can be annoying and may often be dropped for computations.)

Now we have gotten ahead of ourselves, as we don't yet know that AJ is zero on $Z^p_{rat}(X)$. There are 2 ways to see this:

(*) If $\mathcal{W} \in Z^p(\mathbb{P}^1 \times X)$, and $Z_t := \pi^X_* \{\mathcal{W} \cdot (\{t\} \times X)\}$, assume $Z = Z_0 - Z_\infty$. We have $Z_t \underset{rat}{\equiv} Z_0 \Rightarrow Z_t \underset{hom}{\equiv} Z_0 \Rightarrow AJ(Z_t - Z_0)$ is defined, and gives a map $\mathbb{P}^1 \to J^p(X)$, which (by complex fcn. thy.) is constant. So $AJ(Z) = 0$.

(**) If

$$\begin{tikzcd} \tilde{Y} \arrow[r, two heads, "desing."'] \arrow[rr, bend left, "\jmath"] & Y \arrow[r, phantom, "\subset"] & X \end{tikzcd}$$

and $Z = \jmath_*((f))$, $f \in \mathbb{C}(\tilde{Y})^*$, then taking

$$\Gamma_f = \jmath_*(f^{-1}(\mathbb{R}_-)), \quad \Xi_f = \jmath_*\left(\frac{df}{f}\right), \quad R_f = \jmath_*(\boxed{\log f}),$$

↖ 0 current w/ branch cut along $f^{-1}(\mathbb{R}_-)$

we have $d[R_f] = -(2\pi i)\delta_{\Gamma_f} + \Xi_f$. But we have also $\partial\Gamma_f = Z$ and $d[\Xi_f] = 2\pi i\, \delta_Z$, so we are saying $\theta'(E_Z)$ is a coboundary hence trivial! ($\Rightarrow AJ(Z) = 0$).

* * *

We turn next to an abstract point of view (over $\mathbb{Q}$ now).

Definition 2: A mixed Hodge complex (MHC) $K^\cdot$ consists of

- a filtered complex $(K^\cdot_{\mathbb{Q}}, W_\cdot)$ of $\mathbb{Q}$-mod
- a bifiltered complex $(K^\cdot_{\mathbb{C}}, F^\cdot, W_\cdot)$ of $\mathbb{C}$-mod
- a filtered isomorphism $(K_{\mathbb{C}}, W_\cdot) \xleftarrow{\simeq} (K^\cdot_{\mathbb{Q}}, W_\cdot) \otimes_{\mathbb{Q}} \mathbb{C}$

such that the differentials on $Gr^W_\cdot K^\cdot_{\mathbb{C}}$ are strictly compatible with $F^\cdot$ and $F^\cdot$ induces pure polarizable $\mathbb{Q}$-HS of weight $\underline{m+r}$ on $H^r(Gr^W_m K^\cdot_{\mathbb{Q}})$. //

From this we get a diagram of complexes

$$\begin{tikzcd}
 & (K^\cdot_{\mathbb{C}}, W_\cdot) & \\
(K^\cdot_{\mathbb{Q}}, W_\cdot) \arrow[ur, "\beta_1"] & & (K^\cdot_{\mathbb{C}}, W_\cdot, F^\cdot) \arrow[ul, "\beta_2"']
\end{tikzcd}$$

Definition 3 : Given a morphism $(M^\cdot, d_M) \xrightarrow[\mu]{} (N^\cdot, d_N)$ of complexes, the cone of μ is the complex

$$\mathrm{Cone}\{M^\cdot \xrightarrow[\mu]{} N^\cdot\} := \begin{cases} M[1]^\cdot \oplus N & \text{with diff'l} \\ D(a,b) = (-d_M a, \mu(a) + d_N b) \end{cases}$$

Exercise : Check that this defines a complex.

(In particular, if $d_N b = 0$, $(0, b)$ is D-closed.)

Now write

$$(8) \qquad \hat{W}_0 K^m := \ker\left\{ W_{-m} K^m \xrightarrow{d} \frac{K^{m+1}}{W_{-(m+1)} K^{m+1}} \right\}$$

Exercise : Check that this gives a subcomplex of $K^\cdot$ (here $K^\cdot$ means $K^\cdot_{\mathbb{C}}$ or $K^\cdot_{\mathbb{Q}}$)

Put

$$(9) \quad \begin{cases} C^\cdot_{\mathcal{H}} := \mathrm{Cone}\left\{ \hat{W}_0 K^\cdot_{\mathbb{Q}} \oplus F^0 \hat{W}_0 K^\cdot_{\mathbb{C}} \xrightarrow{\beta_1 - \beta_2} \hat{W}_0 K^\cdot_{\mathbb{C}} \right\}[-1] \\ \downarrow \\ C^\cdot_{\mathcal{D}} := \mathrm{Cone}\left\{ K^\cdot_{\mathbb{Q}} \oplus F^0 K^\cdot_{\mathbb{C}} \xrightarrow{\beta_1 - \beta_2} K^\cdot_{\mathbb{C}} \right\}[-1] \\ \quad (= K^\cdot_{\mathbb{Q}} \oplus F^0 K^\cdot_{\mathbb{C}} \oplus K^{\cdot - 1}_{\mathbb{C}} \text{ with } D(a,b,c) = (-da, -db, dc + \beta_1 a - \beta_2 b)) \end{cases}$$

for the absolute Hodge and Deligne complexes associated to the MHC $K^\cdot$.

Next, for X some variety (poss. singular or quasi-proj. or both) take $K^\bullet_{\mathbb{Q}}$, $W_\bullet$ etc. s.t.

$$(10)\quad \begin{cases} H^i(K^\bullet_{\mathbb{Q}}) = H^{i+a}(X,\mathbb{Q}(b)) \quad \left(\Leftrightarrow\ K^\bullet_{\mathbb{Q}} \simeq \bigoplus_i \left(H^i(X,\mathbb{Q}(b))[a]\right)[-i]\right) \\ \mathrm{Im}\{H^i(W_jK^\bullet_{\mathbb{Q}}) \to H^i(K^\bullet_{\mathbb{Q}})\} = W_{i+j}H^{i+a}(X,\mathbb{Q}(b)) \\ \mathrm{Im}\{H^i(F^kW_jK^\bullet_{\mathbb{C}}) \to H^i(K^\bullet_{\mathbb{C}})\} = F^kW_{i+j}H^{i+a}(X,\mathbb{C}(b)) . \end{cases}$$

<u>Exercise</u>: We then have $H^i(\hat{W}_0K^\bullet_{\mathbb{Q}}) \cong \mathrm{Im}\{H^i(W_{-i}K^\bullet_{\mathbb{Q}}) \to H^i(K^\bullet_{\mathbb{Q}})\}$
$= W_0H^{i+a}(X,\mathbb{Q}(b))$, etc. //

So what is the cohomology of $C^\bullet_{\mathcal{H}}$ (resp. $C^\bullet_{\mathcal{D}}$)?

In general, if $C^\bullet = \mathrm{Cone}\{A^\bullet \to B^\bullet\}[-1]$, we have a long-exact sequence

$$(11)\quad \begin{array}{l} \cdots \to H^{i-1}(A^\bullet) \to H^{i-1}(B^\bullet) \to H^i(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to \cdots \\ \qquad\qquad\qquad\qquad\quad b \longmapsto (0,b) \\ \qquad\qquad\qquad\qquad\qquad\qquad (a,b) \longmapsto a \end{array}$$

and so a short-exact sequence

$$(12)\quad 0 \to \frac{H^{i-1}(B^\bullet)}{\mathrm{im}\, H^{i-1}(A^\bullet)} \to H^i(C^\bullet) \to \ker\{H^i(A^\bullet) \to H^i(B^\bullet)\} \to 0$$

For $C_{\mathcal{H}}$, writing $H^i = H^{i+a}(X,\mathbb{Q}(b))$ and $H^{i-1} = H^{i+a-1}(X,\mathbb{Q}(b))$,

$$(13)\quad \begin{cases} \mathrm{RHS}(12) = \ker\{W_0H^i_{\mathbb{Q}} \oplus F^0W_0H^i_{\mathbb{C}} \to W_0H^i_{\mathbb{C}}\} \\ \qquad\qquad = \mathrm{Hom}_{MHS}(\mathbb{Q}(0), H^i) \\ \mathrm{LHS}(12) = W_0H^{i-1}_{\mathbb{C}}/\{F^0W_0H^{i-1}_{\mathbb{C}} + W_0H^{i-1}_{\mathbb{Q}}\} = \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), H^{i-1}) . \end{cases}$$

For $C^\bullet_{\mathcal{D}}$,

$$(14)\begin{cases} \text{RHS }(12) = \ker\{H^i_{\mathbb{C}} \oplus F^0 H^i_{\mathbb{Q}} \to H^i_{\mathbb{C}}\} \\ \text{LHS }(12) = H^{i-1}_{\mathbb{C}} \big/ \{F^0 H^{i-1}_{\mathbb{C}} + H^{i-1}_{\mathbb{Q}}\} \end{cases}$$

which do not have the Hom / Ext^1 description if $\begin{cases} H^i \supsetneq W_0 H^i \\ \text{or } H^{i-1} \supsetneq W_0 H^{i-1} \end{cases}$.

(In the context of cycle maps, this will happen when X is quasi-projective, so that Deligne cohomology is unsuitable for those cases.)

Definition 4:

We set $H^{i+a}_{\mathcal{H}}(X, \mathbb{Q}(b)) := H^i(C^\bullet_{\mathcal{H}})$ and $H^{i+a}_{\mathcal{D}}(X, \mathbb{Q}(b)) := H^i(C^\bullet_{\mathcal{D}})$.

absolute Hodge cohomology   Deligne cohomology

Specializing once more to the case of X smooth projective, we take* for our MHC

$$(15)\begin{cases} K^\bullet_{\mathbb{Q}} = C^{a+\bullet}_{top}(X, \mathbb{Q}(b)) & \text{topological cochains} \\ K^\bullet_{\mathbb{C}} = D^{a+\bullet}(X)(b) & \text{currents } (F^\bullet \text{ as usual}) \\ W_j K^\bullet_{\underset{\in \mathbb{Q}\text{ or }\mathbb{C}}{A}} = \begin{cases} K^\bullet_A & \text{if } j \geq a-2b \\ 0 & \text{if } j < a-2b \end{cases} & \text{stupid filtration} \\ \Rightarrow \hat{W}_0 K^m_A = \begin{cases} K^m_A & \text{if } m < 2b-a \\ \ker(d) \subset K^m_A & \text{if } m = 2b-a \\ 0 & \text{if } m > 2b-a \end{cases} \end{cases}$$

Now let $a = 2p$, $b = p$ so that (by $\underset{-(14)}{(13)}$)

$$(16)\qquad 0 \to J^p(X)_{\mathbb{Q}} \to \underset{(\mathcal{H})}{\overset{}{H^{2p}_{\mathcal{D}}}}\,(X, \mathbb{Q}(p)) \to Hg^p(X)_{\mathbb{Q}} \to 0$$

and

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* the prescription for $K^\bullet$ for quasi-proj. X will be given in a later section

(7) $$\begin{cases} C^{\bullet}_{\mathcal{D}} = C^{2p+\bullet}_{top}(X; \mathbb{Q}(p)) \oplus F^p D^{2p+\bullet}(X) \oplus D^{2p-1+\bullet}(X) \\ \text{with differential } D(T, \Omega, R) = (-\partial T, -d[\Omega], d[R] - \Omega + \delta_T) . \end{cases}$$

<u>Definition 5</u> : The <u>Deligne cycle-class map</u>

$$\underset{(\text{or } c_{\mathcal{H}})}{c^p_{\mathcal{D},X}} : CH^p(X) \longrightarrow H^{2p}_{\underset{(\text{or } \mathcal{H})}{\mathcal{D}}}(X, \mathbb{Q}(p))$$

is given by

$$Z \longmapsto ((2\pi i)^p Z_{top}, (2\pi i)^p \delta_Z, 0) . \quad //$$

<u>Remark</u> : cl, AJ, and $c_{\mathcal{D}}$ are all functorial with respect to <u>pullback</u> (under arbitrary morphisms of smooth varieties).

Exercise : Check this for AJ. //

Clearly Definition 5 gives a cocycle in the complex, and it is well-defined by the (**) above (p.5): we have (for $Z = j_*((f))$)

$$D((2\pi i)^p \Gamma_f, (2\pi i)^{p-1} \Xi_f, (2\pi i)^{p-1} R_f) = (-(2\pi i)^p Z_{top}, -(2\pi i)^p \delta_Z, 0) .$$

$\swarrow$ using the calculation of $d[R_f]$ in (**).

We recover our previous cycle maps by composing

(8) $$\begin{tikzcd} & \mathrm{Hom}_{MHS}(\mathbb{Q}(0), H^{2p}(X, \mathbb{Q}(p))) = Hg^p(X)_{\mathbb{Q}} \\ CH^p(X) \arrow[ur, "cl^p_X"] \arrow[r, "c_{\mathcal{D}}"'] & H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p)) \arrow[u] \\ \ker(cl^p_X) \arrow[u, hook] \arrow[r, "AJ^p_X"] & \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), H^{2p-1}(X, \mathbb{Q}(p))) = J^p(X)_{\mathbb{Q}} \arrow[u] \end{tikzcd}$$

For cl^p_X this is clear (or if not, an exercise!). For AJ, take $c_{\mathcal{D}}(Z)$ and modify it by $D((2\pi i)^p \Gamma, \Xi, 0)$ to get $(0, 0, (2\pi i)^p \delta_\Gamma - \Xi)$.

<u>Remark</u> : With $\mathcal{H}$ replacing $\mathcal{D}$, one has a similar diagram for X <u>smooth quasi-projective</u>, and with "motivic cohomology" $\boxed{H^{2p}_{\mathcal{M}}(X, \mathbb{Q}(p))}$ replacing $CH^p(X)$, a similar diagram for singular X.

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A still more abstract construction of absolute Hodge cohomology arises as follows: let $\mathcal{K}^\bullet \in D^bMHS^{**}$ be a complex quasi-isomorphic to $\bigoplus_i H^{i+a}(X)(b)[-i]$,* and notice that the spectral sequence

$$(\varphi) \qquad \mathrm{Ext}^i_{MHS}(\mathbb{Q}(0), H^j\mathcal{K}^\bullet) \underset{i+j=*}{\overset{\text{converges}}{\Longrightarrow}} \mathrm{Ext}^*_{D^bMHS}(\mathbb{Q}(0), \mathcal{K}^\bullet)$$

$$\Longrightarrow \quad 0 \to \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), \underset{H^{a-1}(X)(b)}{\underset{\parallel}{H^{-1}\mathcal{K}^\bullet}}) \to \mathrm{Hom}_{D^bMHS}(\mathbb{Q}(0), \mathcal{K}^\bullet) \to \mathrm{Hom}_{MHS}(\mathbb{Q}(0), \underset{H^a(X)(b)}{\underset{\parallel}{H^0\mathcal{K}^\bullet}}).$$

$\mathrm{Ext}^0 = \mathrm{Hom})$

$$\Longrightarrow \quad \mathrm{Hom}_{D^bMHS}(\mathbb{Q}(0), \mathcal{K}^\bullet) \cong H^a_{\mathcal{H}}(X, \mathbb{Q}(b)).$$

It will be convenient to use this when we define the "higher" cycle-class maps on ker (AJ).

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Finally, the most abstract point of view is that of MHM (mixed Hodge modules) due to Morihiko Saito. Let

$$\mathcal{V} = (\underset{\uparrow_{\mathbb{Q}\text{-local system}}}{\mathbb{V}}, W_\bullet, \mathcal{V}_{\mathcal{O}}, \mathcal{F}^\bullet, \nabla, \iota) \qquad \text{"VMHS"}/\text{cx. manifold } S$$

be a variation of (graded polarized) MHS: the main new ingredient

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** D^b means bounded derived category (derived = invert quasi-isomorphisms, etc.)

* Remark: (For simplicity take $a=b=0$.) Though we won't need this, people who work with $\mathcal{K}^\bullet$ often prefer the following "canonical" choice to $\bigoplus H^i(X)[-i]$: let $\{H_i\}_{i=0}^{d_X}$, $\{\tilde{H}_j\}_{j=0}^{d_X}$ be 2 general configurations of hyperplane sections of X, $H^I = \bigcup_{i\in I} H_i$, etc., and (for $|I|, |J| > 0$) take $C^{I,J}_{H,\tilde{H}}(X) := H^{d_X}(X \setminus H^I, \tilde{H}^J \setminus \tilde{H}^J \cap H^I; \mathbb{Q})$ then put $\mathcal{K}^\ell := \bigoplus_{I,J \text{ s.t. } |I|-|J| = \ell - d_X} C^{I,J}_{H,\tilde{H}}(X)$. Optional Exercise: Check this works.

is that ∇ preserves $\mathbb{W}_\bullet$ and there must be bilinear forms $Q_\ell : Gr^W_\ell \mathbb{W} \otimes Gr^W_\ell \mathbb{W} \to \mathbb{Q}$ s.t. the $Gr^W_\ell \mathcal{V}$ are PVHS; also, $\iota : \mathbb{V}_{\mathbb{C}} \xrightarrow{\cong} \ker(\nabla) \subset \mathcal{V}_{\mathcal{O}}$

Definition 6: $\mathcal{V}$ is admissible $\Leftarrow$ its restriction to every curve $\subset \mathcal{S}$ satisfies

- (i) local monodromies T are quasi-unipotent
- (ii) the relative weight filtration $M(N, W_\cdot)_\cdot$ exists
- (iii) $F^\cdot$ extends to locally free subsheaves $\mathcal{F}^\cdot_e$ of Deligne's canonical extension $\mathcal{V}_e$ s.t. $Gr_{\mathcal{F}^\cdot_e} Gr^{W^e_\cdot}_\cdot \mathcal{V}_e$ are locally free.

(Note that since ∇ preserves $W_\cdot$, so do T and $N := \log(T_{un})$.)

Here $M_\cdot = M(N, W_\cdot)_\cdot$ must satisfy

(a) $N M_\cdot \subset M_{\cdot - 2}$

(b) $M_\cdot$ restricts on each $Gr^W_\ell V$ to the usual weight monodromy filtration (centered about ℓ).

If it exists, it is unique. //

Denote the category of admissible variations of MHS $/\mathcal{S}$ by AVMHS($\mathcal{S}$).

(By a theorem of Steenbrink & Zucker, all VMHS arising from "algebro-geometric families" are admissible.)

MHM($\mathcal{S}$) is an enlargement of AVMHS($\mathcal{S}$), constructed so as to have something which is closed under the basic operations on sheaves. (The point is to allow degeneration inside $\mathcal{S}$, support over proper subvarieties of $\mathcal{S}$, and so on. So the underlying objects $\mathbb{V}$ can't be local systems any more.)

In brief, the idea is to generalize

certain kind of complex of constructible sheaves (in the derived category)

- the filtered $\mathbb{Q}$-local system $(\mathbb{V}, W)$ to filtered $\mathbb{Q}$-perverse sheaf
- the bifiltered locally free sheaves w/ connection $(\mathcal{V}, \mathcal{F}; W, \nabla)$ to bifiltered regular holonomic D-modules
- $\wr$ by the Riemann-Hilbert correspondence.

More precisely, $AVMHS(S) \subset MHM(S)$ are the "smooth" MHM's (with $\mathbb{V}$ a local system); $D^b MHM$ enjoys operations like $f_*, f_!, f^*, f^!$ (and there is an important decomposition theorem that we shall state later).

[e.g. given a smooth morphism $f: X \to Y$ and the "trivial" (rank 1) MHM $\mathbb{Q}_Y(0)$ on Y, $f_* \mathbb{Q}_Y(0) \underset{q-i}{\simeq} \bigoplus \underbrace{R^i f_* \mathbb{Q}}_{(A)V(M)HS} [-i]$.]

So: another version of absolute Hodge cohomology is given by

$$H^a_{AH}(X, \mathbb{Q}(b)) := \mathrm{Ext}^a_{MHM(X)}(\overset{\overset{a_X^*\mathbb{Q}(0),\ a_X : X \to \mathrm{Spec}\,\mathbb{C}}{\shortparallel}}{\mathbb{Q}_X(0)}, \mathbb{Q}_X(b))$$

$$\cong \mathrm{Ext}^a_{D^b PMHS}(\mathbb{Q}(0), \underset{\underset{K^\bullet[-a]}{\wr}}{a_{X*}\mathbb{Q}_X(b)}) \qquad [\text{since } a_X^* \,\&\, a_{X*} \text{ are adjoint}] \tag{20}$$

$$\cong \mathrm{Hom}_{D^b PMHS}(\mathbb{Q}(0), K^\bullet) \qquad [\text{notice this really doesn't require MHM to define}]$$

$$\xrightarrow[\text{forget}]{} \mathrm{Hom}_{D^b MHS}(\mathbb{Q}(0), K^\bullet)$$

$$\cong H^a_{\mathcal{H}}(X, \mathbb{Q}(b)).$$

Exercise: Use remark (i) on p.3 to check "forget" is injective. //

Moreover: there exists a cycle-class map (X smooth quasi-proj.)

$$c_{AH} : CH^p(X) \to H^{2p}_{AH}(X, \mathbb{Q}(p)) \tag{21}$$

defined using MHM. On the surface, the definition is rather simple: if

$$\begin{tikzcd} Z \arrow[d, phantom, "\cap"] & \tilde{Z} \arrow[l, two heads, "\text{resolve}"'] \arrow[dl, "f"] \\ X & \end{tikzcd}$$

is irreducible of codim p, we simply compose the "natural" maps of MHM

$$(22) \qquad \mathbb{Q}_X(0) \underset{\text{“pullback”}}{\longrightarrow} f_*\mathbb{Q}_{\tilde{Z}}(0) \underset{\text{“pushfwd”}}{\longrightarrow} \mathbb{Q}_X(p)[2p]$$

and regard (22) as an element of $\mathrm{Hom}_{(D^b)MHM(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(p)[2p])$ $\cong \mathrm{Ext}^{2p}_{MHM(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(p)) = H^{2p}_{AH}(X, \mathbb{Q}(p))$. I'm not sure I've seen this used to write <u>explicit</u> maps, which certainly <u>can</u> be done for $c_{\mathcal{H}}$ (X quasi-proj.). But knowing the existence of (21) will be handy later for defining one version of higher AJ maps, and for the Griffiths-Green theorem.