2. Cycles modulo algebraic equivalence

Let X be a smooth projective variety over $\mathbb{C}$.

We first recall the story for divisors. As in the proof of Thm. I.B.1, the exponential exact sequence on X gives

$$\begin{array}{ccccccc} 0 \to & \dfrac{H^1(\mathcal{O})}{H^1(X,\mathbb{Z})} & \to & H^1(\mathcal{O}_X^*) & \xrightarrow{cl} & \ker\{H^2(X,\mathbb{Z}) \to H^2(\mathcal{O})\} & \to 0 \\ & \| & & \| & & \| & \\ & J^1(X) & & Pic(X) \ni [L_D] & & Hg^1(X) & \\ & & & \| \quad\quad \uparrow & & & \\ & & & CH^1(X) \ni [D] & & & \end{array}$$

In particular, we have

(1) $$AJ_X^1 : CH^1_{hom}(X) \xrightarrow{\cong} J^1(X),$$

in which (esp. for $\dim_{\mathbb{C}} X = 1$) injectivity is usually referred to as Abel's theorem and surjectivity as Jacobi inversion. Note that trivially

(2) $$(J^1(X) =)\ CH^1_{hom}(X) = CH^1_{alg}(X),$$

since any two points in $J^1(X)$ can be connected by a curve (with a tautological cycle over it). Also, as the Jacobian of a level-one PHS $H^1(X)$, $J^1(X)$ is an abelian variety: by the H-R bilinear relations, the polarization induces a Kähler metric $h(u,v) = -i\,Q(u,\bar{v})$ on $J^1(X)$ with rational Kähler class, and so $J^1(X)$ is projective algebraic by the Kodaira embedding theorem.

Moving on to higher codimension, we begin with the restriction

(3) $$AJ^k_{alg,X} : CH^k_{alg}(X) \to J^k(X)$$

of the Abel-Jacobi map to cycles algebraically equivalent to zero.

The first main point is that Jacobians of Hodge structures of level > 1 are "generically" non-algebraic complex tori.

Exercise: Show this for level/weight 3, of type $\overset{h^{0,3}h^{2,1}\ldots}{(1,1,1,1)} = \underline{h}$.

(a) Given (V, φ, Q) PHS of this type, we may view φ as factoring through a compact 2-torus $T \leq Sp_4(\mathbb{R}) = Aut(V_{\mathbb{R}}, Q)$, $\varphi(z)$ acting as $\bar{z}^3, z^{-1}, z^1, z^3$ on $V^{0,3}, V^{1,2}, V^{2,1}$ resp. $V^{3,0}$. We can also consider $\hat{\varphi}(z)$ defined to (factor through the same T and) have eigenvalues $\bar{z}^{-1}, \bar{z}^{-1}, z, z$ on the same spaces. This defines the HS $\hat{\varphi}$ on $H^1(J^2(V))$, where $J^2(V) = V_{\mathbb{C}} / (F^2 V_{\mathbb{C}} + V_{\mathbb{Z}})$. Show that if it is polarizable, then $M_{\hat{\varphi}} \neq Sp_4$ (i.e. smaller).

(b) Prove that for φ off a proper analytic subset of $D_{\underline{h}}$, $M_{\hat{\varphi}} \cong Sp_4$, and conclude that (by (a)) "most" $\varphi \in D$ have nonalgebraic Jacobian. [You'll need to state/use Kodaira embedding theorem or the equivalent.]

(c) Show in particular that $Sym^3 H^1(E)$ (E = elliptic curve) has nonalgebraic Jacobian if $H^1(E)$ is not CM. //

Theorem 1 (Lieberman): $\mathrm{im}(AJ^k_{alg,X}) =: J^k_{alg}(X)$ is an abelian variety.

Proof: $$CH^k_{alg}(X) = \sum_{\substack{C \text{ curve} \\ W \in Z^k(C \times X) \text{ cycles}}} W_*(Z^1_{hom}(C))$$ [defn. of algebraic equivalence to 0]

$$\Longrightarrow \quad \mathrm{Im}\left(AJ^k_{alg,X}\right) = \sum_{(C,W)} \mathrm{Im}\left(W_*: J^1(C) \to J^k(X)\right).$$

Let $A \overset{\mu}{\hookrightarrow} J^k(X)$ be a maximal abelian subvariety in this image.
If $\xi \in \mathrm{Im}\{W_*: J^1(C) \to J^k(X)\}$ isn't in A, then the image of $\mu \times W_*: A \times J^1(C) \to J^k(X)$ is abelian* and contains A and ξ. ※
So $A = \mathrm{Im}\left(AJ^k_{alg,X}\right)$. □

Now we have some constraints on the image of (3) arising from the Exercise (or rather the statement preceding it). But the next result actually gives a stronger constraint still.

__Theorem 2__: Define $J^k_{hdg}(X) := J^k(H^{2k-1}_{hdg}(X))$, where $H^{2k-1}_{hdg}(X)$ is the largest sub-HS of $H^{2k-1}(X)$ with $H_{hdg}(X)_{\mathbb{C}} \subset H^{k,k-1}(X) \oplus H^{k-1,k}(X)$. Then we have that

$$(4) \qquad J^k_{alg}(X) \subseteq J^k_{hdg}(X).$$

__Remark__: $J^k_{hdg}(X)$ is of course an abelian variety. Equality in (4) is a __conjecture__, though it is easy to establish in examples.

__Proof (of Thm. 2)__: Let $W \subset X \times C$ be an irreducible subvariety of codimension k, with $\pi_X, \pi_C : \tilde{W} \to X, C$ the projections from a desingularization of W to X resp. C. If we put $Z_i := \pi_{X*}\pi_C^*\{p_i\}$,

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* __Exercise__: use Poincaré complete reducibility to show that the image of an abelian variety A, under a ~~homomorphism~~ $A \to J = \underline{\mathbb{C}\text{x. torus}}$, is an abelian variety.

then $Z_1 \underset{alg}{\equiv} Z_2 \implies Z_1 \underset{hom}{\equiv} Z_2$, which can be seen explicitly by setting $\Gamma := \pi_{X_*}\pi_{\mathcal{C}}^*(\overrightarrow{q \cdot p})$ (so that $Z_1 - Z_2 = \partial\Gamma$).

Let $\omega \in \ker(d) \subset F^{d_X-k+1}A^{2d_X-2k+1}(X)$ be a test form for $\int_\Gamma(\cdot) \in J^k(X)$. Noting that $\pi_{\mathcal{C}}$ has fibers of dimension $d_X - k$ generically over $\mathcal{C}$, so $\pi_{\mathcal{C}_*}$ (integration along fibers / push-forward) "eats up" (d_X-k, d_X-k) from the type of a current. Therefore

$\int_\Gamma \omega = \int_{\pi_{X_*}\pi_{\mathcal{C}}^*(\overrightarrow{q \cdot p})} \omega = \int_q^p \pi_{\mathcal{C}_*}\pi_X^*\omega =: \int_q^p \kappa$, with

$\kappa \in \{\ker(d) \subset F^1A^1(\mathcal{C})\} \underset{\substack{(d\text{-regularity}\\ \text{lemma})}}{=} \Omega^1(\mathcal{C})$, and $\kappa = 0$ unless ω has a component of type (d_X-k+1, d_X-k). Equivalently, $\int_\Gamma(\cdot)$ lifts to an element of $F^{k-1}H^{2k-1}(X,\mathbb{C})/F^k$ $\left(\subset \frac{H^{2k-1}_{\mathbb{C}}}{F^kH^{2k-1}_{\mathbb{C}}} \twoheadrightarrow J^k(X)\right)$ hence to $H^{k-1,k}(X) \oplus H^{k,k-1}(X) \subset H^{2k-1}(X,\mathbb{C})$.

But the image of (3) is an abelian variety a fortiori a complex torus, and there is a 1-1 correspondence between sub-$\mathbb{Z}$HS of $H^{2k-1}(X)$ and sub-tori of $J^k(X)$. So the image of (3) must be contained in the Jacobian of a HS contained in $H^{2k-1}_{alg}(X)$. $\square$

If X is projective of odd dimension $2k-1$, we have the hard Lefschetz decomposition (writing "L_*" for cup-product with hyperplane class)

$$H^{2k-1}(X) = \underset{\substack{\parallel \\ \ker(L_*)}}{H^{2k-1}_{prim}(X)} \oplus L_*(H^{2k-3}(X)).$$

and one may have that $H^{2k-1}_{hdg}(X) \subset L_*(H^{2k-3}(X))$ (esp. if H^{2k-1}_{prim} is irreducible). For X in a Lefschetz pencil on $\mathbb{X}$, we often have image$(H^{2k-1}(\mathbb{X})) \subset L_*(H^{2k-3}(X))$, and so the following often gives a way to arrange this.

<u>Proposition 1</u>: Let $X \overset{\iota}{\hookrightarrow} \mathbb{X}$ be very general in a Lefschetz pencil on $\mathbb{X}$ (sm. proj. of dim $2k$), and assume $\text{level}(H^{2k-1}(X)) > \underbrace{1, \text{level}(H^{2k-1}(\mathbb{X}))}$. Then $H^{2k-1}_{hdg}(X) \subset \iota^* H^{2k-1}(\mathbb{X})$ and $J^k_{alg}(X) \subset \iota^* J^k(\mathbb{X})$.

<u>Proof</u>: If $\{X_s\}_{s \in \mathbb{P}^1}$ is the pencil (of hyperplane sections, $X_s = H_s \cdot \mathbb{X}$, in which singular fibers have one ODP each), let $B = X_0 \cap X_\infty$ be the base locus and $\mathcal{X}$ the blow-up of $\mathbb{X}$ along B. We have a diagram

{disc. locus}

$$\begin{tikzcd} & \mathcal{X}^* \arrow[r, hook] \arrow[d, "\pi"] & \mathcal{X} \arrow[r, "\beta"] \arrow[d, "\bar{\pi}"] & \mathbb{X} \\ \mathbb{P}^1 \setminus \Sigma = & U \arrow[r, hook] & \mathbb{P}^1 & \end{tikzcd}$$

and $X = X_{t \in U}$ is a very general fiber. It is known that $\Gamma :=$ image of
$$\rho : \pi_1(U) \to \text{Aut}(H^{2k-1}(X_t, \mathbb{Q}), Q)$$
acts irreducibly on $H_{var} := (H_{fix})^\perp \subset H^{2k-1}(X_t, \mathbb{Q})$, and $H_{fix} = \iota^* H^{2k-1}(\mathbb{X})$. Since the MTG of $H^{2k-1}(X_t, \mathbb{Q})$ is generic, $H^{2k-1}_{hdg}(X)$ extends to a level 1 sub VHS of $R^{2k-1}\pi_* \mathbb{Q} \cong (\iota^*) H^{2k-1}(\mathbb{X}) \oplus H_{var}$. Assumptions imply that H_{var} has level > 1. Since H_{var} has no sub VHS, $H^{2k-1}_{hdg}(X) \subset \iota^* H^{2k-1}(\mathbb{X})$, and we are done by Theorem 2. $\square$

Now we turn to the "quotient" or "reduced" AJ map on the Griffiths group $\mathrm{Griff}^k(X) := Z^k_{hom}(X)/Z^k_{alg}(X) = CH^k_{hom}(X)/CH^k_{alg}(X)$, that is,

$$(5) \qquad \overline{AJ}^k_X : \mathrm{Griff}^k(X) \to J^k(X)/J^k_{alg}(X) \twoheadrightarrow J^k(X)/J^k_{hdg}(X) =: \bar{J}^k(X).$$

Proposition 2: $\mathrm{Griff}^k(X)$ (and thus $\mathrm{im}(\overline{AJ}^k_X)$) is of most countable.

Proof: Given a cycle $\check{Z}/L$ on X/K ($L \supset K$ both f.g. $/\bar{\mathbb{Q}}$) consider the $\bar{\mathbb{Q}}$-spread

$$(6) \qquad \begin{array}{ccc} \mathcal{X} & \supset & X \times \mathcal{S}_{t_0} \\ \downarrow & & \downarrow \\ \mathcal{S} & \supset & \mathcal{S}_{t_0} \\ \downarrow & & \downarrow \\ \tilde{J} & \ni & \{t_0\} \end{array} \quad , \quad \mathfrak{Z} \in \mathrm{Griff}^k(\mathcal{X}) \qquad (\text{all def'd } /\bar{\mathbb{Q}}),$$

where $\bar{\mathbb{Q}}(\mathcal{T}) = K$ and $\bar{\mathbb{Q}}(\mathcal{S}) = L$. Now there are only countably many situations like (6), and all cycles $Z \in \mathrm{Griff}^k(X/\mathbb{C})$ occur as some fiber $\mathfrak{Z}_s$, $s \in \mathcal{S}_{t_0}(\mathbb{C})$, in some situation "(6)". But all the fibers $\mathfrak{Z}_s$, $s \in \mathcal{S}_{t_0}(\mathbb{C})$, in any given situation "(6)" are algebraically equivalent as there is always a (chain of) curves connecting any $s, s' \in \mathcal{S}_{t_0}(\mathbb{C})$! $\square$

Remark (on "cardinalities"): Note that $(\otimes \mathbb{Q})$ $\mathrm{Griff}^k(X)$ and its image could still be a countably ∞-dim'l $\mathbb{Q}$-vector space. Contrast this to the image of AJ^k_{alg}: any complex torus (e.g. elliptic curve $/\mathbb{C}$) $\otimes \mathbb{Q}$ is of uncountably-infinite dimension as a $\mathbb{Q}$-vector space. But $\mathrm{im}(AJ^k_{alg})$ is still parametrized by a finite dim'l algebraic variety. Later we will speak of cycle groups not being "pro-representable", which means this is impossible. So there are 3 levels of "bigness" here. One should think of $\mathrm{Griff}^k(X)$ as the discrete/totally-disconnected part, and CH^k_{alg} as the "continuous" part, of $CH^k_{hom}(X)$. //

Corollary: Jacobi inversion fails for $CH^k_{hom}(X)$ if $H^{2k-1}(X)_{\mathbb{C}} \neq H^{k,k-1} \oplus H^{k-1,k}$.

We now turn to an example due to B. Harris, which will be our first computation of an AJ map.

Consider the Fermat quartic curve $\mathcal{F} = \overline{\{x^4+y^4=1\}} \subset \mathbb{P}^2$. By the degree-genus formula, it has genus 3, with holomorphic forms given by Griffiths's residue formula:

$$\frac{P(x,y)\, dx \wedge dy}{\underbrace{x^4+y^4-1}_{=F}} = \frac{P\,dx}{F_y} \wedge \frac{dF}{F} \xrightarrow{\mathrm{Res}} \frac{P\,dx}{4y^3}, \quad P = 1, x, y. \quad \deg \leq \deg(\mathcal{F})-2-1 = 1$$

After normalizing, we get the basis of $\Omega^1(\mathcal{F})$

(7) $\omega_1 = \frac{1-i}{4b}\frac{dx}{y^2}$, $\omega_2 = \frac{1}{2b'}\frac{dx}{y^3}$, $\omega_3 = \frac{1-i}{4b}\frac{x\,dx}{y^3}$,

where $b := \int_0^1 \frac{dt}{(1-t^4)^{1/2}}$, $b' = \sqrt{2}\, b = \int_0^1 \frac{dt}{(1-t^4)^{3/4}}$.

The reason for this normalization is that we have three morphisms from $\mathcal{F}$ to the elliptic curve given by $E := \overline{\{ v^2 = 1-u^4 \}}$, given by

(8) $\pi_1(x,y) := (x, y^2)$, $\pi_2(x,y) := \left(\frac{-1+i}{\sqrt{2}} \frac{y}{x} , \frac{1}{x^2} \right)$, $\pi_3(x,y) := (-y, x^2)$,

and the form "dz" $:= \frac{1-i}{4b} \frac{du}{v} \in \Omega^1(E)$ has $\pi_j^*(dz) = \omega_j$ $(j=1,2,3)$.

Exercises: (i) Check this.

Moreover, dz has periods 1 & i over cycles generating $H_1(E,\mathbb{Z})$.

(ii) Check this too! [So $E \cong \mathbb{C}/\mathbb{Z}\langle 1, i \rangle$]

Since $\pi = (\pi_1, \pi_2, \pi_3) : \mathcal{F} \to E^{\times 3}$ induces an isomorphism ($\Rightarrow$ surjection) $\pi^* : H^1(E^{\times 3}, \mathbb{C}) \to H^1(\mathcal{F}, \mathbb{C})$, we have that $\pi_* : H_1(\mathcal{F}, \mathbb{Z}) \to H_1(E^{\times 3}, \mathbb{Z})$ is an injection. (If $\pi_j(\gamma) \overset{\text{hom}}{\equiv} 0$ $(\forall j)$, then $0 = \int_{\pi_j(\gamma)} dz = \int_\gamma \pi_j^* dz = \int_\gamma \omega_j$ $(\forall j)$, so $\gamma \underset{\text{hom}}{\equiv} 0$.)

Recall the Abel map

$$\phi : \mathcal{F} \longrightarrow J^1(\mathcal{F}) = \frac{\{\Omega^1(\mathcal{F})\}^\vee}{H_1(\mathcal{F}, \mathbb{Z})} \cong \mathbb{C}^3/\Lambda$$

given by $p \longmapsto AJ(p - o_{\mathcal{F}}) := \int_o^p (\cdot) = \left(\int_o^p \omega_1, \int_o^p \omega_2, \int_o^p \omega_3 \right)$ where $o_{\mathcal{F}} = (1,0)$.* But the RHS clearly $= \left(\pi_1(p), \pi_2(p), \pi_3(p) \right)$, and so

(9)
$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\pi"] \arrow[dr, "\phi"'] & & E^{\times 3} \\ & J^1(\mathcal{F}) \arrow[ur, "\rho"'] & \end{tikzcd}$$
ρ = isogeny (induced by $(\pi^*)^\vee$)

commutes.

* I take the origin of E_j to be the π_j images of $o_{\mathcal{F}}$.

Now we consider the <u>Ceresa cycle</u>

(10) $$Z_{\sigma_{\mathcal{F}}} := \phi(\sigma_{\mathcal{F}}) - \phi(\sigma_{\mathcal{F}})^- \in Z^2_{hom}(J^1(\sigma_{\mathcal{F}})),$$

where $(\cdot)^-$ means to apply the involution $\underline{u} \mapsto -\underline{u}$ on $J^1(\mathcal{F})$.

(If $\mathcal{F}$ were hyperelliptic, this would just be zero; but, as we shall see, it isn't.) Note that this involution acts as the identity on $H^4(J^1(\sigma_{\mathcal{F}}))$, which is why $Z_{\sigma_{\mathcal{F}}} \underset{hom}{\equiv} 0$. Our question is: <u>is it algebraically equivalent to zero</u>?

One way to show it is <u>not</u> $\underset{alg}{\equiv} 0$ is to show it has nonzero image under

(11) $$\overline{AJ}^2_{J^1(\sigma_{\mathcal{F}})} : \mathrm{Griff}^2(J^1(\sigma_{\mathcal{F}})) \to \overline{J}^2(J^1(\sigma_{\mathcal{F}})),$$

but it turns out to be easier to work with

(12) $$\overline{AJ}^2_{E^{\times 3}} : \mathrm{Griff}^2(E^{\times 3}) \to \overline{J}^2(E^{\times 3})$$

and $\rho(Z_{\sigma_{\mathcal{F}}}) = \pi(\sigma_{\mathcal{F}}) - \pi(\sigma_{\mathcal{F}})^-$. Write dz_j ($j=1,2,3$) for the copies of dz on each factor of $E^{\times 3}$.

<u>Exercise</u>: Since E is CM, $H^3_{alg}(E^{\times 3})_{\mathbb{C}}$ is the $\perp$ complement of

$$\mathbb{C}\langle dz_1 \wedge dz_2 \wedge dz_3,\ d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle.$$

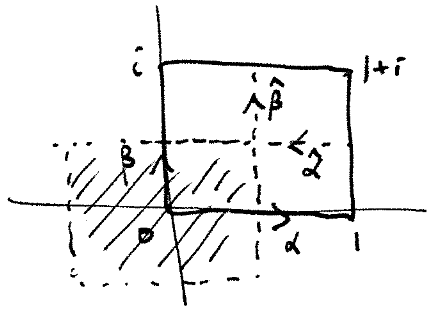
Conclude that $\overline{J}^2(E^{\times 3}) = \dfrac{\mathbb{C}\langle dz_1 \wedge dz_2 \wedge dz_3\rangle^\vee}{\mathbb{Z}\langle 1, i\rangle} \cong \mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$. //

So, writing Γ for a 3-chain on $E^{\times 3}$ with $\partial\Gamma = \pi(\sigma_{\mathcal{F}}) - \pi(\sigma_{\mathcal{F}})^-$, if we can show that

(13) $$\int_\Gamma dz_1 \wedge dz_2 \wedge dz_3 \notin \mathbb{Z} \oplus i\mathbb{Z},$$

then we win.

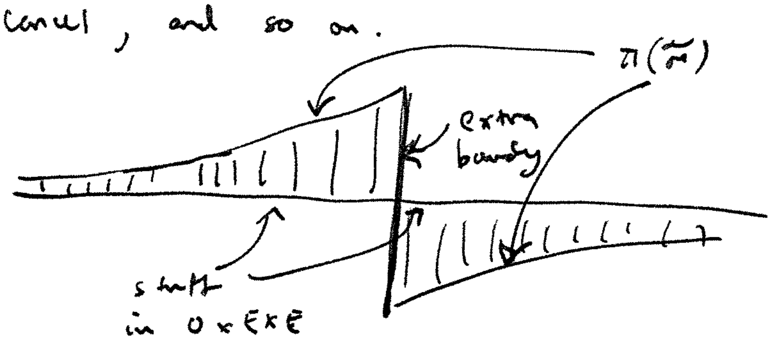
Suppose first that we want to draw a chain Γ_+ with boundary $\pi(\tilde{\sigma}) - \{\text{stuff supported on } \{0\}\times E\times E \cup E\times\{0\}\times E \cup E\times E\times\{0\}\}$. To do this, draw cuts $\hat{\alpha}$ & $\hat{\beta}$ on E:



and write $z_j := \int_0^* dz_j$ on the shaded region with jumps along $\hat{\alpha}$ & $\hat{\beta}$, and $\overrightarrow{0.p}$ ($p \in E$) to denote the shortest path from the origin to p (which changes as we pass thru. the cuts!). Then we set

$$\Gamma_+ = \left\{ \left(\overrightarrow{0.\pi_1(f)}, \pi_2(f), \pi_3(f)\right) \,\middle|\, f \in \tilde{\sigma} \right\}$$
$$+ \left\{ \left(\text{“}\beta\text{”}, \overrightarrow{0.\pi_2(f)}, \pi_3(f)\right) \,\middle|\, f \in \pi_1^{-1}(\hat{\alpha}) \right\} + \left\{ \left(\text{“}\alpha\text{”}, \overrightarrow{0.\pi_2(f)}, \pi_3(f)\right) \,\middle|\, f \in \pi_1^{-1}(\hat{\beta}) \right\}$$
$$+ \left\{ \left(\beta, \beta, \overrightarrow{0.\pi_3(f)}\right) \,\middle|\, f \in \pi_1^{-1}(\hat{\alpha}) \cap \pi_2^{-1}(\hat{\alpha}) \right\} + 3 \text{ more terms}$$

where the idea is that ∂ of the first term gives $\pi(\tilde{\sigma}) - \{\text{stuff on } \{0\}\times E\times E\}$ if we stay away from $\pi_1^{-1}(\text{cuts})$, but at the cuts the jump in how we draw $\overrightarrow{0.\pi_1(f)}$ creates an extra boundary term, which the next two terms' boundaries cancel, and so on.



Integrating $dz_1 \wedge dz_2 \wedge dz_3$ over Γ_+ yields $\int_{\tilde{\alpha}} \pi^*(z_1\, dz_2 \wedge dz_3)$ which is zero by type. The next two terms yield (since $\int_\beta dz_1 = i$, $\int_\alpha dz_1 = 1$)

$$i \int_{\pi_1^{-1}(\hat\alpha)} \underbrace{\mathfrak{z}_2\, \omega_3}_{\left(\text{i.e. } z_2\, dz_3 \text{ pulled back to } \sigma_\mu\right)} + \int_{\pi_1^{-1}(\hat\beta)} \mathfrak{z}_2\, \omega_3$$

where $\mathfrak{z}_j := z_j \circ \pi_j$, and the final terms yield

$$-\sum_{p \in \pi_1^{-1}(\hat\alpha) \cap \pi_2^{-1}(\hat\alpha)} \mathfrak{z}_3(p) + i \sum_{p \in \pi_1^{-1}(\hat\beta) \cap \pi_2^{-1}(\hat\alpha)} \mathfrak{z}_3(p) + i \sum_{p \in \pi_1^{-1}(\hat\alpha) \cap \pi_2^{-1}(\hat\beta)} \mathfrak{z}_3(p) + \sum_{p \in \pi_1^{-1}(\hat\beta) \cap \pi_2^{-1}(\hat\beta)} \mathfrak{z}_3(p).$$

Exercise: If $\int_{\gamma;p} \omega_2 \circ \omega_3 := \int_\gamma \underbrace{\left(\int_p^* \omega_2\right)}_{\substack{\text{gives function} \\ \text{(starting from } \underline{0} \text{ at } p) \\ \text{on } \gamma}} \omega_3$ (for $\gamma \in \pi_1(\sigma_F)$, $p \in \gamma$)

show that the final terms serve the function of correcting the top two terms to $-i \int_{\pi_1^{-1}(\alpha);0} \omega_2 \circ \omega_3 + \int_{\pi_1^{-1}(\beta);0} \omega_2 \circ \omega_3$.

Next we do the same thing for $\pi(\sigma_F)^-$ with $\Gamma_- = (\Gamma_+)^-$, and observe that the integral $-\int_{\Gamma_-} dz_1 \wedge dz_2 \wedge dz_3$ simply doubles the terms above.

Finally, Γ will be $\Gamma_+ - \Gamma_- + \{\underbrace{\text{stuff supported on } O\times E\times E \cup E\times O\times E \cup E\times E\times O}_{\text{over which } dz_1\wedge dz_2\wedge dz_3 \text{ integrates to zero}}\}$

$$\Rightarrow \int_\Gamma dz_1\wedge dz_2 \wedge dz_3 = -2i \int_{\pi_1^{-1}(\alpha)} \omega_2 \circ \omega_3 + 2\int_{\pi_1^{-1}(\beta)} \omega_2 \circ \omega_3$$

$$\underset{\mathbb{Z}\langle 1,i\rangle}{\equiv} 8(1+i) \int_{(1,0)}^{(i,0)} \omega_2 \circ \omega_3 = 8(1+i) \int_{(0,1)}^{(1,0)} \omega_1 \circ \omega_2$$

using automorphism $(x,y) \mapsto \left(\frac{y}{x\sqrt{2}}, \frac{1}{ix}\right)$?

$$= 8(1+i) \int_0^1 \frac{(1-i)}{4b} \frac{dx}{\sqrt{1-x^4}} \circ \frac{1}{2b'} \frac{dx}{(1-x^4)^{3/4}}$$

$$= \frac{4\int_0^1 \frac{dt}{\sqrt{1-t^4}} \circ \frac{dt}{(1-t^4)^{3/4}}}{\left(\int_0^1 \frac{dt}{\sqrt{1-t^4}}\right)\left(\int_0^1 \frac{dt}{(1-t^4)^{3/4}}\right)} =: \kappa \quad \left(\in \mathbb{C}/_{\mathbb{Z}\langle 1, i\rangle}\right).$$

Of course, $\kappa \in \mathbb{R}$ so we win if $\kappa \notin \mathbb{Z}$. Harris numerically computes that $\kappa \approx 1.24178\ldots$, and so we get that

$Z_{\mathfrak{F}}$ is not algebraically equivalent to zero in $J'(\mathfrak{F})$.

Unfortunately, one doesn't know whether κ is irrational*, and so one does not get the

Theorem 3 (Bloch): $Z_{\mathfrak{F}}$ is of infinite order in $\mathrm{Griff}^2(J'(\bar{\mathfrak{F}}))$.

which is proved instead using the ℓ-adic AJ map ($X/\mathbb{Q}$)

$$AJ^2_{\ell, X} : CH^2_{hom}(X) \longrightarrow H^1_{cont}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), H^3_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell(2))).$$

[Formally speaking, this map is easy to construct: one has a cycle map $CH^2(X) \to H^4_{et}(X, \mathbb{Z}_\ell(2))$ and a spectral sequence (Hochschild-Serre) $H^p_{cont}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), H^q_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell(2))) \Rightarrow H^{p+q}_{et}(X, \mathbb{Z}_\ell(2))$,

and
$$CH^2_{hom}(X) = \ker\left(CH^2(X) \to H^0(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), H^4_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell(2)))\right)$$
$$= \ker\left(CH^2(X) \to H^4_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell(2))\right). \quad]$$

What is interesting about Bloch's proof is that AJ_ℓ of $Z_{\mathfrak{F}}$, once shown to be nonzero, is almost "automatically" of infinite order (see §4 of his paper "Algebraic cycles & values of L-functions").

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* though this does follow from Theorem 3 if the Bloch-Beilinson conjectures (which we'll discuss later) hold.