3. Infinite generation of the Griffiths group

In this section we shall more systematically investigate the Ceresa cycle

(1) $$Z_C := \phi(C) - \phi(C)^- \in Z^{g-1}_{hom}(J^1(C)),$$

for C a genus g curve (esp. a very general one).

But there are some well-definedness issues to address first.

For one thing,

(2) $$\begin{aligned} \phi : C &\longrightarrow J^1(C) \\ p &\longmapsto AJ(p-p_0) \end{aligned}$$

depends on the choice of p_0. Of course, since the various "$\phi(C)$" to which choices of p_0 give rise are all algebraically equivalent (b/c parametrized by C), Z_C is well-defined in $\mathrm{Griff}^{g-1}(J(C))$. In fact if C is defined $/k$, then this shows that $\langle Z_C \rangle \in \mathrm{Griff}^{g-1}$ is $\mathrm{Gal}(\overline{K}/K)$-invariant.

For very general C, Sp_{2g} is the M-T group of $H^1(C)$ (and $J^1(C)$), hence of $H^{2g-3}(J^1(C)) \cong H^3(J^1(C)) \cong \Lambda^3 H^1(C)$. (Use the fact that symplectic transvections, arising from nodal degenerations via the Picard-Lefschetz formula, generate the full symplectic group: this gives that geometric monodromy $= Sp_{2g}$, which forces $MTG = Sp_{2g}$.)

Since sub-HS are subrepresentations of the MT group, and $\Lambda^3 st \cong st \oplus \text{irred.}$ as a representation of Sp_{2g}, we get that

$$(3) \qquad H^{2g-3}(J'(C)) \cong \underbrace{H^{2g-3}_{prim}(J'(C))}_{\text{level 3, irred}} \oplus \underbrace{H^1(C)(2-g)}_{\substack{\text{level 1} \\ (\leftrightarrow st)}}$$

as HS. It follows that $H^{2g-3}_{hdg}(J'(C)) \cong H^1(C)(2-g)$, and so reduced AJ takes the form

$$(4) \qquad \overline{AJ}^{g-1}_{J'(C)} : Griff^{g-1}(J'(C)) \to J^{g-1}(H^{2g-3}_{prim}(J'(C))).$$

We conclude that $\nu_C := \overline{AJ}^{g-1}(Z_C)$ yields a section of the bundle $\mathcal{J}_g = \coprod_{C \in \mathcal{M}_g} J^{g-1}(H^{2g-3}_{prim}(J'(C)))$ of complex tori over (a Zariski open subset of) $\mathcal{M}_g$.*

With this in mind, we state our first

Theorem 1 (Ceresa): For C very general, of genus ≥ 3 (in particular, nonhyperelliptic!), ν_C is nontorsion. Hence, Z_C is of infinite order in $Griff^{g-1}(J'(C))$.

Proof of Theorem 1 in the genus 3 case: Suppose we have a family of

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* even this statement requires qualification (!): there are "two" Jacobians of the universal curve over $\mathcal{M}_g$, and this statement is only true for one of them (the one that is "quadratically twisted" over the hyperelliptic locus).

1-cycles $Z_s \underset{\text{hom}}{\equiv} 0$ on 3-folds X_s with $H^3_{hdg}(X_s) = L_* H^1(X_s)$ generically (L_* = cup product w/ polarizing class), and a family of chains Γ_s with $\partial \Gamma_s = Z_s$, s.t.

(5) $$\Gamma_0 = 0 = Z_0 .$$

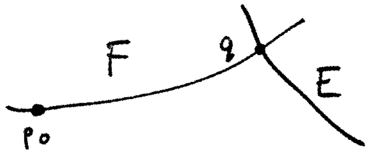
Writing $\{\underline{\omega}_s^i\}$ for a basis of $F^2 H^3_{prim}(X_s, \mathbb{C})$,

(6) $$\underset{(\forall s)}{\overline{AJ}(Z_s) = 0} \implies \underset{(\forall i,s)}{\int_{\Gamma_s} \underline{\omega}_s^i = \int_{\gamma_s} \underline{\omega}_s^i} \text{ for some family of cycles } \gamma_s \implies \underset{(\forall s)}{\int_{\Gamma_s} \underline{\omega}_s^i = 0} .$$

Thus for Γ_s, Z_s satisfying (5),

(7) $$\underset{(\text{for some } i_0, s_0)}{\int_{\Gamma_s} \omega^{i_0}_{s_0} \neq 0} \implies \overline{AJ}(Z_s) \neq 0 \text{ for very general } s .$$

To use (7), degenerate



(8) $$C \rightsquigarrow C_0 = E \cup F , \quad \begin{cases} E = \text{ell. curve} \\ F = \text{smooth genus } 2 \\ E \cap F = \{q\} \end{cases}$$

$\implies J(C_0) = E \times J(F)$ with

$\omega_{C_0} = \omega_E + \omega_F$ as polarizing (1,1) forms.

Write $j : F \to F$ for the hyperelliptic involution, and assume $p_0 = j(p_0)$.

Consider $\Psi = \mu \wedge (\omega_E - \omega_F)$, $\mu \in \Omega^1(J(F))$. Then

$$\Psi \wedge \omega_{C_0} = \mu \wedge (\omega_E - \omega_F) \wedge (\omega_E + \omega_F) \underset{\uparrow}{=} 0 \implies \Psi \text{ primitive}$$

$\implies [\Psi] \in F^2 H^3_{prim}(J(C_0), \mathbb{C})$. (degree-2 forms commute rather than anticommute.)

Under the degeneration (8), $\phi(C)$ becomes (up to translation)

$\phi(C_0) = \{0\} \times \phi(F) + E \times \{\phi(q)\}$, and so Z_C specializes to

$$(9) \qquad Z_{C_0} = E \times \{\phi(q)\} - E \times \{\underset{\substack{\shortparallel \\ \phi(j(q))}}{-\phi(q)}\} .$$

We take q near to p_0 and $\Gamma = E \times \overrightarrow{-\phi(q).\phi(q)}$, so that $Z, \Gamma \to 0$ as $q \to p_0$. (That is, in (5)-(7), $s=0$ corresponds to C_0 with $p_0 = q$; and $s = s_0$ corresponds to C_0 with $q \neq p_0$.) Now compute

$$(10) \qquad \int_\Gamma \Psi = \int_{E \times \overrightarrow{-\phi(q).\phi(q)}} \mu \wedge (\omega_E - \omega_F) = \int_{-\phi(q)}^{\phi(q)} \mu \neq 0$$

for some choice of μ (and any $q \neq p_0$). Applying (7), we are done. $\square$

Our next result is

Theorem 2 (Nori): For C very general of genus 3, $\mathrm{Griff}^2(J^1(C))$ (and $\mathrm{image}(\overline{AJ}^2_{J^1(C)})$) is infinitely generated.

Proof: (This is the long one; the other ones are short.)

$\boxed{\text{Step 1}}$ We begin by explaining a diagram (actually a fiber square)

$$(11) \qquad \begin{tikzcd} \text{universal Jacobian} & J(\tilde{C}_3) \arrow[r] \arrow[d] & \tilde{A}_3 \arrow[d] & \text{universal abelian 3fold} \\ & T_3 \arrow[r, "2:1"] & h_3 \supset \mathcal{H} = \text{hyperelliptic locus} & \\ & \text{Torelli space} & \text{Siegel upper } \tfrac{1}{2}\text{-space} & \end{tikzcd}$$

universal genus 3 curve

$\circlearrowleft \; Sp_6(\mathbb{Z})$-equivariant

First of all: T_3 is the moduli space of $\overset{(\cong \text{ classes of})}{\{}$genus 3 curves with a choice of symplectic basis of $H_1(C)\}$; denote points by $(C,[\underline{\alpha},\underline{\beta}])$.

Points of $J(\tilde{\mathcal{C}}_3)$ are $(C,[\underline{\alpha},\underline{\beta}],u)$ where $u \in \Omega^1(C)^\vee / H_1(C,\mathbb{Z})$.

The map from $T_3 \to h_3$ is defined by $(C,[\underline{\alpha}\underline{\beta}]) \longmapsto \int_{\underline{\beta}} \underline{\omega}(\underline{\alpha}) =: \tau(C,[\underline{\alpha},\underline{\beta}])$, where $\underline{\omega}(\underline{\alpha})$ is the basis of $\Omega^1(C)$ with $\int_{\underline{\alpha}} \underline{\omega}(\underline{\alpha}) =$ identity matrix.

This is a 2:1 mapping (since $\underline{\omega}(-\underline{\alpha}) = -\underline{\omega}(\underline{\alpha}) \Rightarrow \tau(C,[-\underline{\alpha},-\underline{\beta}]) = \tau(C,[\underline{\alpha},\underline{\beta}])$) ramified over the hyperelliptic locus.

Exercise: Why ramified there?

The top horizontal arrow of (1), at least away from the hyperelliptic locus, is defined by $(C,[\underline{\alpha},\underline{\beta}],u) \longmapsto (\tau(C,[\underline{\alpha},\underline{\beta}]), ev_{\underline{\omega}(\underline{\alpha})} u)$, where $ev_{\underline{\omega}} : J^1(C) \xrightarrow{\cong} \mathbb{C}^g / \mathbb{Z}\langle \mathbb{1}, \tau(C,[\underline{\alpha},\underline{\beta}])\rangle = A_\tau$ evaluates the functional u against the basis $\underline{\omega}$.

For the action of $Sp_6(\mathbb{Z})$, $\gamma = \overset{3\times 3 \text{ matrices}}{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}$ sends $(\overset{3\times 3}{\tau}, \overset{3\times 1}{z}) \longmapsto ((C\tau+D)^{-1}(A\tau+B), (C\tau+D)^{-1}z)$, and $(C,[\alpha,\beta],u) \mapsto (C,[\gamma\alpha,\gamma\beta],u)$.

Exercise: Show that this "intertwines" the top arrow of (1) (i.e. prove the equivariance).

The key point in all of this is the action of "$-id$" $(\in Sp_6(\mathbb{Z}))$. This gives the involution of T_3 over h_3, flipping it about its hyperelliptic locus. Its action on $J(\tilde{\mathcal{C}}_3)$ and $\tilde{\mathcal{C}}_3$ over T_3 intertwines the Abel map

$\phi : C \to J(C)$ modulo $\equiv_{alg}$, and hence pulls the Ceresa cycle back to itself (modulo $\equiv_{alg}$). But the quotient of $J(\tilde{\mathcal{C}}_3)$ by $-id$ does NOT yield $\tilde{\mathcal{A}}_3$, but rather a different family $\tilde{\mathcal{A}}_3'$ with order-2 monodromy about $\mathcal{H}$!* (This arises from the fact that $-id$ sends $z \mapsto -z$ on each A_τ.) There is a different involution of $J(\tilde{\mathcal{C}}_3)$ given by $(C, [\alpha, \beta], u) \mapsto (C, [-\alpha, -\beta], -u)$ that yields $\tilde{\mathcal{A}}_3$ as a quotient, but this sends the Ceresa cycle to *minus* itself, which means that it **dies** upon pushforward to $\tilde{\mathcal{A}}_3$. So in a way, *you can think of the universal Z_C as "corresponding" to the 2:1 branched cover of $\mathfrak{h}_3$.*

Step 2 Take the quotient of (11) by $\Gamma(\ell) = \ker\{Sp_6(\mathbb{Z}) \to Sp_6(\mathbb{Z}/\ell\mathbb{Z})\}$. This gives

$$(12) \qquad \begin{tikzcd} J(\mathcal{C}(\ell)) \arrow[r] \arrow[d] & \mathcal{A}(\ell) \arrow[d] & \\ \mathcal{M}_3(\ell) \arrow[r] & X_3(\ell) & \circlearrowleft Sp_6(\mathbb{Z}/\ell\mathbb{Z}) \end{tikzcd}$$

$\mathcal{M}_3(\ell)$: moduli space of genus 3 curves with full level-ℓ structure

$X_3(\ell)$: moduli space (locally symm. variety) of 3-dim'l PPAVs with full level-ℓ structure

and we can take the direct limit over ℓ and pass to the generic fibers to get

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* $\tilde{\mathcal{A}}_3'$ and $\tilde{\mathcal{A}}_3$ pull back to the same family over (a Zar. op. subset of) T_3.

$$\begin{tikzcd} J(C) \arrow[r] \arrow[d] & A \arrow[d] \\ \mathrm{spec}\,\mathbb{E} \arrow[r] & \mathrm{Spec}\,\mathbb{F} \end{tikzcd} \quad \circlearrowleft Sp_6(\hat{\mathbb{Z}}) \supset Sp_6(\mathbb{Q}) \tag{13}$$

where $\mathbb{E}$ and $\mathbb{F}$ are the $\varinjlim_{l}$'s of function fields of $\mathcal{M}_3(l)$ resp. $\mathcal{X}_3(l)$. Here the isomorphy of $\tilde{\mathcal{A}}_3'$ & $\tilde{\mathcal{A}}_3$ upon pullback to T_3 translates to an isomorphism $\underline{f : J(C) \xrightarrow[\cong]{} A_{\mathbb{E}}}$.

$\boxed{\text{Step 3}}$ Any degree 2 extension $\tilde{\mathbb{F}} : \mathbb{F}$ corresponds to a ramified 2:1 cover of some $X_3(l)$. Write $D(\tilde{\mathbb{F}})$ for the preimage of the ramification locus on $\mathfrak{h}_3$. Note that $D(\mathbb{E}) = \mathcal{H}$, and that $Gal(\tilde{\mathbb{F}}/\mathbb{F})$ acts on the Ceresa cycle

$$\Theta := f_{\tilde{\mathbb{F}}*}(Z_C) \in Griff^2(A_{\tilde{\mathbb{F}}}) \tag{14}$$

through the character $Gal(\tilde{\mathbb{F}}/\mathbb{F}) \twoheadrightarrow Gal(\mathbb{E}/\mathbb{F}) \xrightarrow[\cong]{} \{\pm 1\}$.

$\boxed{\text{Step 4}}$ The action of $Sp_6(\mathbb{Q})$ in (13) on $\mathbb{F}$ can be described as follows. Since the center $\{\pm 1\}$ acts trivially, it extends to

$$\begin{tikzcd} \rho_1 : & GSp_6(\mathbb{Q}) \arrow[r] & \mathrm{Aut}(\mathbb{F}) \\ \Big(& \mathbb{Q}^* \arrow[u, phantom, sloped, "\subset"] \arrow[r, mapsto] & \{1\} \arrow[u, phantom, sloped, "\in"] & \Big) \end{tikzcd} ;$$

and for $g \in M_6(\mathbb{Z}) \cap GSp_6(\mathbb{Q})$ (which together w/ $\mathbb{Q}^*$ generate $GSp_6(\mathbb{Z})$), we get (for $\underline{\text{some}}$ l & a) a diagram

$$\begin{tikzcd} \mathcal{A}(al) \arrow[r, twoheadrightarrow] \arrow[d] & \mathcal{A}(l) \arrow[d] \\ X_3(al) \arrow[r, twoheadrightarrow] & X_3(l) \end{tikzcd} \tag{15}$$

fiberwise isogenies

induced by action of g on $\mathfrak{h}_3$

hence (passing to the limit of generic pt.) the diagram

(16)
$$\begin{tikzcd} A \arrow[r, "\rho_2(g)"'] \arrow[d] & A \arrow[d] \\ \mathrm{Spec}\,\mathbb{F} \arrow[r, "\rho_1(g)"'] & \mathrm{Spec}\,\mathbb{F} \end{tikzcd}$$

and a cycle

(17)
$$\theta(g) := \rho_2(g)_* \theta \in \mathrm{Griff}^2(A_{\bar{\mathbb{F}}}).$$

If we let $\mathbb{E}(g) : \mathbb{F}$ be the degree-two extension corresponding to $g \cdot \mathcal{H} \subset \mathfrak{h}_3$, then $\mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F})$ acts on $\theta(g)$ thru the character

(18)
$$\chi_g : \mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F}) \twoheadrightarrow \mathrm{Gal}(\mathbb{E}(g)/\mathbb{F}) \xrightarrow{\cong} \{\pm 1\}.$$

Clearly $g\mathcal{H} \neq g'\mathcal{H} \Rightarrow \chi_g \neq \chi_{g'} \Rightarrow \theta(g)$ & $\theta(g')$ independent$/\mathbb{Q}$.

Step 5 Observe that the stabilizer of $\mathcal{H}$ in $Sp_6(\mathbb{Q})$ is $Sp_6(\mathbb{Z})$. (See Totaro or Nori for the easy proof.) Since this has infinitely many cosets in $Sp_6(\mathbb{Q})$, we get (countably) infinitely many characters χ_g, hence (countably) infinitely many independent $\theta(g)$'s. To see these in Griff^2 of a very general abelian 3-fold$/\mathbb{C}$, just take an embedding $\mathbb{F} \hookrightarrow \mathbb{C}$. □

We mention one more (very recent) result; with proof similar to Nori's:

Theorem 3 (Totaro): For a very general 3-dim'l PPAV $A/\mathbb{C}$, $\mathrm{Griff}^2(A)/\ell$ is infinitely generated for every prime number ℓ.