4. Admissible normal functions and their invariants

We first begin with the notion of a *normal function* (minus admissibility), which is a section of a bundle of intermediate Jacobians (recall these are complex tori) associated to a VHS of odd weight; this section is usually assumed to satisfy a transversality assumption which is called *quasi-horizontality*.

Let $\mathcal{V}$ be a polarized ($\mathbb{Z}$-)VHS of weight (-1) over a complex manifold $\mathcal{S}$, with underlying (flat) local system $\mathbb{V}_{\mathbb{Z}}$ and associated intermediate Jacobian bundle $\mathcal{J}(\mathcal{V})$. Form the complexes*

$$(1)\quad \begin{cases} C^\bullet := \mathcal{V} \xrightarrow{\nabla} \Omega^1_{\mathcal{S}} \otimes \mathcal{V} \xrightarrow{\nabla} \Omega^2_{\mathcal{S}} \otimes \mathcal{V} \xrightarrow{\nabla} \cdots \\ F^p C^\bullet := \mathcal{F}^p \mathcal{V} \xrightarrow{\nabla} \Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{p-1}\mathcal{V} \xrightarrow{\nabla} \Omega^2_{\mathcal{S}} \otimes \mathcal{F}^{p-2}\mathcal{V} \xrightarrow{\nabla} \cdots \\ Gr^p_{\mathcal{F}} C^\bullet := Gr^p_{\mathcal{F}} \mathcal{V} \xrightarrow{\overline{\nabla}} \Omega^1_{\mathcal{S}} \otimes Gr^{p-1}_{\mathcal{F}} \mathcal{V} \xrightarrow{\overline{\nabla}} \Omega^2_{\mathcal{S}} \otimes Gr^{p-2}_{\mathcal{F}} \mathcal{V} \xrightarrow{\overline{\nabla}} \cdots \end{cases}$$

of sheaves on $\mathcal{S}$, noting that $\overline{\nabla}$ is $\mathcal{O}_{\mathcal{S}}$-linear.

Exercise: Check the assertion that the "graded Gauß-Manin connection" $\overline{\nabla}$ is $\mathcal{O}_{\mathcal{S}}$-linear.

This produces an exact sequence

$$(2)\qquad 0 \to F^0 C^\bullet \oplus \mathbb{V}_{\mathbb{Z}} \to C^\bullet \to \frac{C^\bullet}{F^0 C^\bullet \oplus \mathbb{V}} \to 0 .$$

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* $\mathcal{V}$ also denotes the sheaf of sections of the corresponding holo. vector bundle

Note that $\dfrac{C^0}{F^0C^0 \oplus \mathbb{V}_{\mathbb{Z}}} = \dfrac{\mathcal{V}}{\mathcal{F}^0\mathcal{V} \oplus \mathbb{V}_{\mathbb{Z}}}$ is the sheaf of sections of the Jacobian bundle $\mathcal{J}(\mathcal{V})$.

<u>Definition 1</u>: A section is <u>quasi-horizontal</u> if ∇ of its local liftings to $\mathcal{V}$ lie in $\Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{-1}\mathcal{V}$. //

So the sheaf of quasi-horizontal sections of $\mathcal{J}(\mathcal{V})$ is just

$$(3) \qquad \mathcal{J}_{hor}(\mathcal{V}) := \underbrace{\mathcal{H}^0}_{\text{hypercohomology sheaf}}\left(\frac{C^\bullet}{F^0C^\bullet \oplus \mathbb{V}_{\mathbb{Z}}}\right) = \ker\left\{\frac{\mathcal{V}}{\mathcal{F}^0\mathcal{V} \oplus \mathbb{V}_{\mathbb{Z}}} \to \Omega^1_{\mathcal{S}} \otimes \mathcal{V}/\mathcal{F}^{-1}\mathcal{V}\right\}.$$

<u>Definition 2</u>: The $\mathcal{J}(\mathcal{V})$-valued <u>normal functions</u> over $\mathcal{S}$ are the elements of

$$(4) \qquad NF_{\mathcal{S}}(\mathcal{V}) := \mathbb{H}^0\left(\mathcal{S}, \frac{C^\bullet}{F^0C^\bullet \oplus \mathbb{V}_{\mathbb{Z}}}\right) = \Gamma(\mathcal{S}, \mathcal{J}_{hor}(\mathcal{V})). \qquad //$$

This is the same thing as a variation of mixed Hodge structure $/\mathcal{S}$ of the form

$$(5) \qquad 0 \to \mathcal{V} \to \mathcal{E} \longrightarrow \mathbb{Z}_{\mathcal{S}}(0) \to 0$$

<u>Exercise</u>: Check this – namely, that Griffiths transversality of the VMHS translates to the quasi-horizontality condition.

Now we want a technical condition that holds for "geometric" NF's and which guarantees existence of LMHS. Recall that for an arbitrary VMHS we had the condition of <u>admissibility</u> (cf §I.C.1). Here is what this looks like for a VMHS of the form (5).

Definition 3 (Admissible normal functions, $ANF_{\mathcal{S}}(\mathcal{V}) \subset NF_{\mathcal{S}}(\mathcal{V})$):

It will suffice to take $\mathcal{S} = (\Delta^*)^k$ (coords $z_1, \ldots, z_k$), since this is a condition on the behavior at the boundary.

(i) First assume the monodromies $\{T_j\}$ about the $\{z_j = 0\}$ are unipotent on $\mathbb{V}$. Then they and their logarithms $N_j = \log T_j$ extend to $\mathbb{E}$. Write $\tilde{\mathbb{E}} = e^{-\frac{1}{2\pi i}\sum_j \log(z_j) N_j} E_{(\mathbb{Q})}$ ("untwisted" local system on Δ^k) and $\mathcal{E}_e = \tilde{\mathbb{E}} \otimes \mathcal{O}_{\Delta^k}$, and $M_\bullet^{(j)} = W(N_j)_\bullet$. Then $\mathcal{E}$ is admissible $\iff$

(a) $\exists$ lift $\nu_{\mathbb{Q}} \in (\tilde{\mathbb{E}})_0$ of $1 \in \mathbb{Q}(0)$ s.t. $N_j \nu_{\mathbb{Q}} \in M_{-2}^{(j)}(\tilde{\mathbb{V}})_0$ $(\forall j)$

and (b) $\exists$ lift $\nu_F(s) \in \Gamma(\Delta^k, \mathcal{E}_e)$ of $1 \in \mathbb{Q}_{\mathcal{S}}(0)$ s.t. $\nu_F|_{(\Delta^*)^k} \in \Gamma(\mathcal{S}, F^0\mathcal{E})$.

(ii) In general, $\exists$ minimal finite cover $\rho : (\Delta^*)^k \to (\Delta^*)^k$ (sending $\underline{z} \mapsto \underline{z}^{\underline{\mu}}$) s.t. the $T_i^{\mu_i}$ are unipotent. $\mathcal{E}$ is admissible $\iff$ $\rho^*\mathcal{E}$ satisfies (i). //

Exercise: Check this definition agrees with the previous (more general) definition. //

There are two types of invariants associated to normal functions: those defined "on the interior", and those defined "on the boundary". For the first type, consider the connecting homomorphism

$$(6) \qquad NF_{\mathcal{S}}(\mathcal{V}) \to \mathbb{H}^1(\mathcal{S}, F^0 C^\bullet \oplus \mathbb{V}_{\mathbb{Z}}) = \mathbb{H}^1(F^0 C^\bullet) \oplus H^1(\mathcal{S}, \mathbb{V}_{\mathbb{Z}})$$

associated to (2). The hypercohomology spectral sequence gives

$$(7) \qquad 0 \to H^1(\mathcal{S}, \mathcal{H}^0_\nabla(F^0 C^\bullet)) \to \mathbb{H}^1(F^0 C^\bullet) \xrightarrow[\text{("edge homomorphism")}]{\epsilon} \Gamma(\mathcal{S}, \mathcal{H}^1_\nabla(F^0 C^\bullet)) \to 0$$

Composing (6) with ϵ and restricting to $ANF_{\mathcal{S}}(\mathcal{V})$ gives

$$(8) \qquad ANF_{\mathcal{S}}(\mathcal{V}) \xrightarrow{(\delta, [\cdot])} \Gamma(\mathcal{S}, \mathcal{H}^1_\nabla(F^0 C^\bullet)) \oplus H^1(\mathcal{S}, \mathbb{V}_{\mathbb{Z}}).$$

Definition 4: δ is the infinitesimal invariant and $[\cdot]$ the topological invariant.

Theorem 1: Assume $H^0(\mathcal{S}, \mathbb{V}) = \{0\}$. Then $[\cdot]$ is injective "$\otimes \mathbb{Q}$". *

(TFP)

We will need the Theorem of the fixed part for AVMHS (of Steenbrink & Zucker), which states that the maximal constant subsystem of the local system underlying an AVMHS, underlies a constant sub-AVMHS.

Proof of Thm. 1: Any $\nu \in ANF_{\mathcal{S}}(\mathcal{V})_{\mathbb{Q}}$ is equivalent to an AVMHS extension

(9) $$0 \to \mathcal{V} \to \mathcal{E}_\nu \longrightarrow \mathbb{Q}_{\mathcal{S}}(0) \to 0.$$

If $[\nu] = 0 \in H^1(\mathcal{S}, \mathbb{V}) \cong \mathrm{Ext}^1_{\pi_1(\mathcal{S})}(\mathbb{Q}_{\mathcal{S}}, \mathbb{V})$, then $\mathbb{E} = \mathbb{V} \oplus \mathbb{Q}$. Since $H^0(\mathcal{S}, \mathbb{V}) = 0$, $H^0(\mathcal{S}, \mathbb{E}) \cong \mathbb{Q}$. By TFP, this "$\mathbb{Q}$" underlies a constant sub-AVMHS of $\mathcal{E}_\nu$, necessarily of rank 1 and thus of type $(0,0)$. It $\therefore$ splits (9) and forces $\nu = 0$. $\square$

Theorem 2: If $\mathbb{H}^0_\nabla(F^0 C^\cdot) = 0$, then δ is injective "$\otimes \mathbb{Q}$" (on $NF_{\mathcal{S}}(\mathcal{V})$, in fact).

Proof: The assumption gives that ϵ is an isomorphism, so we only need to show $NF_{\mathcal{S}}(\mathcal{V})_{\mathbb{Q}} \hookrightarrow \mathbb{H}^1(F^0 \mathbb{C}^\cdot)$. Modifying (2) to

$$0 \to F^0 C^\cdot \to C^\cdot/\mathbb{V}_{\mathbb{Z}} \to \frac{C^\cdot}{F^0 C^\cdot \oplus \mathbb{V}_{\mathbb{Z}}} \to 0$$

we see that it is enough to show $\mathbb{H}^0(C^\cdot/\mathbb{V}) = 0$.

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* this means that anything in the kernel is a torsion normal function.

Now we claim that $H^0(S, \mathbb{V}) = 0$. If not, then $H^0(S,\mathbb{V})$ underlies a constant sub-VHS, and given $v \in H^0(S,\mathbb{V})\setminus\{0\}$ we can decompose it into Hodge components $\sum v^{j,-j}$, with $v^{j_0,-j_0} \neq 0$ for some $j_0 \geq 0$. This gives a nonzero global section of $\ker \nabla \cap \mathcal{F}^0\mathcal{V} = \mathcal{H}^0_\nabla(F^0C^\bullet)$, which was assumed zero, a contradiction.

$$\text{So } \quad \mathbb{H}^0(C^\bullet/\mathbb{W}) = H^0(S, \mathbb{V}_{\mathbb{C}}/\mathbb{V}) = \cancel{H^0(S,\mathbb{V})} \otimes_{\mathbb{Q}} \mathbb{C}/\mathbb{Q} = 0, \text{ done.}$$ □

Remark: Since $\mathcal{H}^0_\nabla(F^0 C^\bullet) = 0 \Rightarrow H^0(S,\mathbb{V}) = 0$, we could have used this assumption in Theorem 1 too. //

Let

$$(10) \qquad \mathcal{H}^k_\nabla(j) := \mathcal{H}^k_\nabla(\mathrm{Gr}^j_F C^\bullet).$$

Corollary: Assume $\mathcal{H}^0_\nabla(j)$ and $\mathcal{H}^1_\nabla(j)$ vanish for $j \geq 0$. Then $\mathrm{ANF}_T(f^*\mathbb{V}) = \{0\}$ for all étale maps $T \xrightarrow{f} S$.

Proof: Use the spectral sequence

$$(11) \qquad E_1^{p,q} := \begin{cases} \mathcal{H}^{p+q}(p), & p \geq 0 \\ 0, & p < 0 \end{cases} \underset{p+q=*}{\Longrightarrow} \mathcal{H}^*_\nabla(F^0C^\bullet)$$

to see that the assumption $\Rightarrow \mathcal{H}^0_\nabla(F^0C^\bullet) = 0 = \mathcal{H}^1_\nabla(F^0C^\bullet)$. The result follows then from Theorem 2. □

We'll apply these results in the sections that follow

We now turn briefly to the "on the boundary" invariants.

The natural question concerns <u>limits</u> of normal functions: where these land, obstructions to their existence (if any), etc. Write $j: S \hookrightarrow \bar{S}$ and $\iota_x : \{x\} \hookrightarrow \bar{S}$. Since

(12) $$ANF_S(\mathcal{V})_{\mathbb{Q}} \cong \mathrm{Ext}^1_{AVMHS(S)}(\mathbb{Q}(0), \mathcal{V})$$ and $AVMHS(S) \subset MHM(S)$,

we can "canonically extend" $\mathcal{V}$ to $j_{!*}\mathcal{V} \in D^b MHM(\bar{S})$ ⊛ then restrict to x via $\iota_x^* j_{!*}\mathcal{V} \in D^b MHM(\{x\}) = D^b AMHS \subset D^b MHS$, to define the middle row of

(13)
$$\begin{tikzcd}
 & 0 \\
 & \mathrm{Hom}_{MHS}(\mathbb{Q}(0), H^1 K^\bullet) \arrow[u] \\
ANF_S(\mathcal{V}) \arrow[ur, "\mathrm{sing}_x"] \arrow[r, "(\iota_x^* Rj_*)^{Hdg}"'] & \mathrm{Ext}^1_{D^b MHS}(\mathbb{Q}(0), K^\bullet) \arrow[u] \\
\ker(\mathrm{sing}_x) \arrow[u, hook] \arrow[r, "\lim_x"'] & \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), H^0 K^\bullet) \arrow[u] \\
 & 0 \arrow[u]
\end{tikzcd}$$
$(K^\bullet := \iota_x^* j_{!*}\mathcal{V})$

<u>Definition 5</u>: The (rational) <u>singularity</u> of $\nu \in ANF_S(\mathcal{V})$ at $x \in \bar{S}$ is $\mathrm{sing}_x(\nu)$ (as defined by the diagram); and the <u>limit</u> of ν at x, defined if it is nonsingular there, is $\lim_x(\nu)$ (also as defined by (13)).

Note the formal analogy to cl^k_X and AJ^k_X (where we had a similar diagram in §I.C.1, with C_D in the middle).

For $S = (\Delta^*)^k$ and $\mathbb{V}$ unipotent, it turns out that

(14) $$K^\bullet \simeq \left\{ \mathcal{V}_{lim} \xrightarrow{\oplus N_j} \oplus_j N_j \mathcal{V}_{lim}(-1) \xrightarrow{\oplus N_i} \oplus_{i<j} N_i N_j \mathcal{V}_{lim}(-2) \to \cdots \right\},$$

⊛ Here $j_{!*} := \mathrm{Im}\{Rj_! \to Rj_*\}$ is the perverse middle (or intermediate) extension functor; in this case $j_{!*}\mathcal{V}$ is a subcomplex "retaining" $Rj_*\mathcal{V}$.

so that $\lim_x$ maps into $J(\underbrace{\cap_j \ker N_j}_{\subset \mathcal{V}_{\lim}})$. If $k=1$, we see that $H^1K^\bullet = \{0\}$, so that there are no singularities (rationally) in codimension 1. If $k=2$, the group detecting singularities (in codim. 2) is

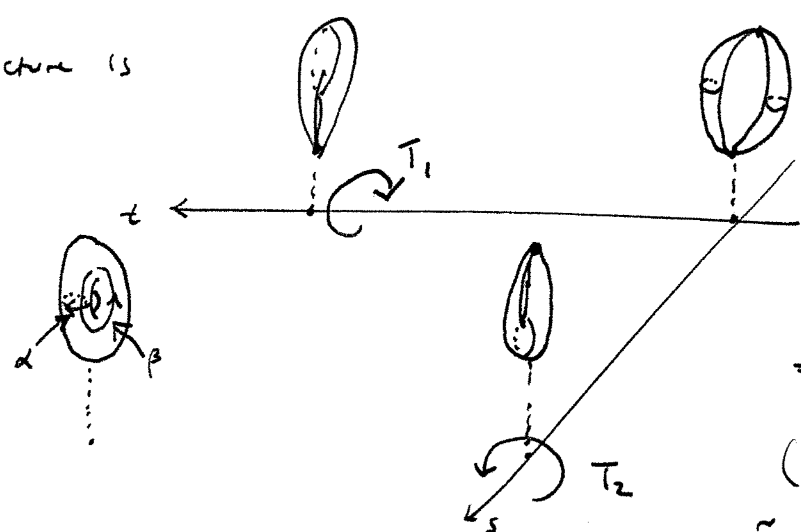
(15) $$\mathrm{Hom}_{MHS}\left(\mathbb{Q}(0),\ \overbrace{\frac{\ker(N_1)\cap \mathrm{im}(N_2)}{N_2(\ker N_1)}}^{H^1K^\bullet}\right) = \left(\frac{\ker(N_1)\cap\mathrm{im}(N_2)}{N_2(\ker N_1)}\right)^{(-1,-1)}_{\mathbb{Q}}.$$

Exercise: Check this using ((14)) (easy).

We conclude with an example of a singularity in codim. 2. Consider the smooth family of elliptic curves over $(\Delta^*)^2$ given by

(16) $$C_{s,t} := \overline{\{x^2+y^2+sx^2y^2+t=0\}} \subset \mathbb{P}^1\times\mathbb{P}^1 .$$

The picture is



$T_1 = T_2 \ :\ \alpha\mapsto\alpha,\ \beta\mapsto\beta+\alpha$

$\Rightarrow\ N_1 = N_2 \ :\ \beta\mapsto\alpha\mapsto 0$.

(In $\mathcal{V}_{\lim}$, β is replaced by $\tilde{\beta}$ = specialization to 0 of $\beta - \frac{\log(st)}{2\pi i}\alpha$, but the N_i's are the same.)

Our complex $K^\bullet$ is simply (restricting to rational stuff)

(17) $$\begin{array}{ccc} \mathbb{Q}\langle\alpha,\tilde{\beta}\rangle & \xrightarrow{N_1\oplus N_2} & \mathbb{Q}\langle\alpha\rangle\oplus\mathbb{Q}\langle\alpha\rangle \\ \alpha & \longmapsto & (0,0) \\ \beta & \longmapsto & (\alpha,\alpha). \end{array}$$

So far I have said nothing about computing sing_x.

Given a cycle family $Z_{s,t}$, you take $\underline{\underline{\Gamma_{s,t}}}$ with $\partial\Gamma_{s,t} = Z_{s,t}$.

This will be multivalued, and you look at its monodromy;

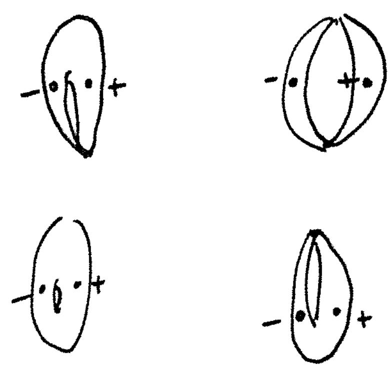
observe that $(T_j - I)\,\Gamma_{s,t}$ will be a cycle, and N_j = polynomial in $(T_j - I)$ (here it actually $\underline{\text{is}}$ $T_j - I$), so $N_j\Gamma_{s,t} \in N_j V_{lim}$ is defined. (In fact, this amounts to computing $N_j v_\alpha$ in Defn. 3(i)(a).)

In this way, you get an element of $H^1 K^\bullet$ (which is rational and of type $(-1,-1)$).

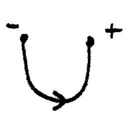
Consider the cycle

$$(18) \qquad Z_{s,t} := \left(i\sqrt{\frac{1+t}{1+s}}\,,\,1\right) - \left(-i\sqrt{\frac{1+t}{1+s}}\,,\,1\right)$$

which looks like this in the above picture:



So that if we draw $\Gamma_{s,t}$ like this  then $\begin{cases} N_1\Gamma = 0 \\ N_2\Gamma = \alpha \end{cases}$

while if $\Gamma_{s,t}$ is , then $\begin{cases} N_1\Gamma = \alpha \\ N_2\Gamma = 0 \end{cases}$. Neither $(0,\alpha)$ nor $(\alpha,0)$ is in the image of (17)! So we get a nontrivial singularity class for the normal function $AJ(Z_{s,t}) =: \nu$. In fact, since by "nontrivial" we mean nontorsion, this implies at once ν itself is nontorsion!

<u>Remark</u>: We should mention the following connection between the topological invariant $[\cdot]$ and the singularity invariants sing_x described above: a more refined version of $[\cdot]$ maps $ANF_{\mathcal{S}}(\mathcal{V})$ into $IH^1(\bar{\mathcal{S}}, \mathbb{V})$ rather than $H^1(\mathcal{S}, \mathbb{V})$. When $\mathcal{S}$ is a <u>curve</u>, we have $IH^1(\bar{\mathcal{S}}, \mathbb{V}) = H^1(\bar{\mathcal{S}}, \jmath_* \mathbb{V})$ $(\jmath : \mathcal{S} \hookrightarrow \bar{\mathcal{S}})$ because $\jmath_{!*} = \jmath_*$ (<u>not</u> $R\jmath_*$); note that by the Leray spectral sequence for $\jmath$, we have

$$0 \to H^1(\bar{\mathcal{S}}, \jmath_* \mathbb{V}) \hookrightarrow \underset{\substack{\| \\ H^1(\mathcal{S}, \mathbb{V})}}{H^1(\bar{\mathcal{S}}, R\jmath_* \mathbb{V})} \twoheadrightarrow H^0(\bar{\mathcal{S}}, R^1 \jmath_* \mathbb{V}) \to 0 .$$

In the curve case, we will say something about how this refinement comes about at the end of the next section.

The point is now that the singularity invariants factor through this refined topological invariant:

$$\begin{array}{ccc} ANF_{\mathcal{S}}(\mathcal{V})_{\mathbb{Q}} & \xrightarrow{[\cdot]} & IH^1(\bar{\mathcal{S}}, \mathbb{V}) \\ \downarrow \mathrm{sing}_x & & \downarrow \\ \mathrm{Hom}_{MHS}(\mathbb{Q}(0), H^1 \iota_x^* \jmath_{!*} \mathcal{V}) & = & IH^1_x(\bar{\mathcal{S}}, \mathbb{V}) \end{array}$$

$= \underset{\text{cohom. at } x}{\text{local intersection}}$

commutes. In the curve case, local IH^1 is zero <u>rationally</u>, but we can work integrally:

$$\begin{array}{ccc} ANF_{\mathcal{S}}(\mathcal{V}) & \xrightarrow{[\cdot]} & \ker\{H^1(\mathcal{S}, \mathbb{V}_{\mathbb{Z}}) \oplus H^1(\bar{\mathcal{S}}, \jmath_* \mathbb{V}) \to H^1(\mathcal{S}, \mathbb{V})\} \\ \downarrow \mathrm{sing}_{x,\mathbb{Z}} & & \downarrow \\ G_x := \boxed{\dfrac{\{(T_x - I)\mathbb{V}_{s_0(\mathbb{Q})}\} \cap \mathbb{V}_{s_0, \mathbb{Z}}}{(T_x - I)\mathbb{V}_{s_0, \mathbb{Z}}}} & = & \ker\{H^1(\underbrace{\Delta^*}_{\text{punctured disk at } x}, \mathbb{V}_{\mathbb{Z}}) \to H^1(\Delta^*, \mathbb{V})\} \end{array}$$

(monodromy about x)

Note that G_x is a <u>finite group</u>, and $\mathrm{sing}_{x,\mathbb{Z}}$ sends $\nu(s) = AJ(Z_s)$, $Z_s = \partial \Gamma_s$, to "$(T_x - I)\Gamma_{s_0}$" — you compute the monodromy of the bounding chain.